Applied Mathematics and Computation 284 (2016) 149–161

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Some novel optimal eighth order derivative-free root solvers and their basins of attraction

Janak Raj Sharma*, Himani Arora

Department of Mathematics, Sant Longowal Institute of Engineering and Technology, Longowal 148106, Punjab, India

ARTICLE INFO

Keywords:
Nonlinear equations
Multipoint iterative methods
Derivative free methods
Optimal convergence order
Attraction basins

ABSTRACT

We present two families of derivative-free methods with eighth order convergence for solving nonlinear equations. Each method of the families requires four function evaluations per full iteration, that means, the families are optimal in the sense of the hypothesis of Kung–Traub (1974). Computational results and comparison (including CPU time) with existing methods confirm the efficient and robust character of new methods. Moreover, the presented basins of attraction also confirm equal or better performance of the methods as compared to other established methods in literature.

1. Introduction

The problem of finding solution of the nonlinear equation $f(x) = 0$ is a fundamental task in scientific computation. The well-known method for this purpose is the quadratically convergent Newton's method [1], possessing the following iterative scheme

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}, \quad k = 0, 1, 2, \ldots, \tag{1}$$

where x_0 is the initial guess to a root. The method requires two evaluations per iteration, namely f and f'.

In many practical situations it is preferable to avoid the calculation of derivative $f'(x)$ of the function $f(x)$. In such situations the approximation of $f'(x)$ by employing the values of $f(x)$ at appropriate points is more suitable. For example, a basic derivative free iterative method is the Traub–Steffensen method [2], which also converges quadratically and follows the scheme

$$x_{i+1} = x_i - \frac{f(x_i)}{f[w_i, x_i]}, \tag{2}$$

where $w_i = x_i + \beta f(x_i)$, $\beta \neq 0$ is any real constant and $f[\cdot, \cdot]$ is the first order divided difference. This method is obtained by replacing the derivative $f'(x_i)$ in (1) by its divided difference approximation $f[w_i, x_i]$. For $\beta = 1$, the method (2) reduces to the well-known Steffensen's method [3].

Using various techniques to approximate the derivatives, many higher order derivative-free multipoint methods have been proposed in literature, see for example [4–14]. In particular, using Padé approximants of different degree for the approximation of derivatives, Cordero et al. [13] have proposed multipoint methods of optimal fourth, eighth and sixteenth

* Corresponding author. Tel.: +91 1672253256; fax: +91 1672280057.
E-mail addresses: jrshira@yahoo.co.in (J.R. Sharma), arorahimani362@gmail.com (H. Arora).

http://dx.doi.org/10.1016/j.amc.2016.02.054

order of convergence in the sense of Kung–Traub hypothesis [4]. According to this hypothesis multipoint methods without memory based on n function evaluations have optimal order 2^{n-1}. Such methods are usually called optimal methods. The fourth order method proposed in [13] is given by

$$\begin{cases} y_i = x_i - \frac{f(x_i)}{f[w_i,x_i]}, \\ x_{i+1} = y_i - \frac{f[w_i,x_i]}{f[y_i,w_i]f[y_i,x_i]} f(y_i), \end{cases} \tag{3}$$

where $w_i = x_i + f(x_i)$. The method is obtained by replacing the derivative $f'(y_i)$ in the second step of the following scheme

$$\begin{cases} y_i = x_i - \frac{f(x_i)}{f[w_i,x_i]}, \\ x_{i+1} = y_i - \frac{f(y_i)}{f'(y_i)}, \end{cases} \tag{4}$$

by $m'(y_i) = \frac{f[w_i,x_i]}{f[y_i,w_i]f[y_i,x_i]}$, where $m(t) = \frac{a_1+a_2(t-y_i)}{1+a_3(t-y_i)}$ is the first degree Padé approximant, wherein a_1, a_2, a_3 are the real parameters obtained from the following conditions

$$m(x_i) = f(x_i), \ m(w_i) = f(w_i) \text{ and } m(y_i) = f(y_i).$$

Similarly, using the second and third degree Padé approximants, the corresponding eighth and sixteenth order methods have been presented in [13].

Another important technique to remove derivative is by using the Newtonian interpolatory polynomials explained in [7]. The fourth order scheme presented in this paper is given by

$$\begin{cases} y_i = x_i - \frac{f(x_i)}{f[w_i,x_i]}, \\ x_{i+1} = y_i - \frac{f(y_i)}{f[y_i,w_i]+f[y_i,x_i]-f[w_i,x_i]}. \end{cases} \tag{5}$$

This scheme is obtained by approximating the derivative $f'(y_i)$ in the second step of (4) by $N_2'(y_i)$, where $N_2(t) = f(y_i) + f[y_i, x_i](t-y_i) + f[y_i, x_i, w_i](t-y_i)(t-x_i)$, (wherein $f[\cdot,\cdot,\cdot]$ is the second order divided difference) such that

$$N_2(x_i) = f(x_i), \quad N_2(w_i) = f(w_i) \quad \text{and} \quad N_2(y_i) = f(y_i). \tag{6}$$

Using nth degree interpolatory polynomial, a general family of optimal order 2^n has also been proposed in [7].

In this paper, our aim is to develop optimal eighth order methods by applying Padé approximation and Newtonian interpolation techniques. Consequently, we present two families of methods which have simple structure and possess optimal eighth order convergence according to Kung–Traub hypothesis [4]. The paper is divided into five sections and is organized as follows. In Section 2, using the idea of Padé approximations, the first family is developed and its convergence analysis is discussed. Then, based on the idea of Newtonian interpolation, the second family with its convergence analysis is presented in Section 3. The theoretical results proved in Sections 2 and 3 are verified in Section 4 by considering various numerical examples. Here, a comparison of the new methods with the existing methods is also performed. In Section 5, the basins of attractors for the new methods and some existing eighth order methods are presented. Section 6 contains the concluding remarks.

2. The eighth order family-I

Let us begin with the following three-step iterative scheme

$$\begin{cases} y_i = x_i - \frac{f(x_i)}{f[w_i,x_i]}, \\ z_i = \Phi_4(x_i, y_i), \\ x_{i+1} = z_i - \frac{f(z_i)}{f'(z_i)}. \end{cases} \tag{7}$$

Here, $\Phi_4(x_i, y_i)$ is any optimal fourth order scheme with the base as Traub–Steffensen iteration. The above iterative scheme uses the evaluation of derivative $f'(z_i)$. In order to make this a derivative free scheme, we use some suitable approximation of the derivative $f'(z_i)$. We will obtain this approximation by considering the following Padé approximant of first degree

$$m(t) = \frac{a_1 + a_2(t-z_i)}{1 + a_3(t-z_i)}, \tag{8}$$

where a_1, a_2 and a_3 are determined by the following conditions

$$m(x_i) = f(x_i), \ m(y_i) = f(y_i) \text{ and } m(z_i) = f(z_i). \tag{9}$$

From (8) and first condition of (9), we obtain

$$a_1 = f(z_i). \tag{10}$$

Substituting the value of a_1 into (8) and then using the last two conditions of (9), some simple calculations lead to

$$a_2 = f[z_i, y_i] - f(y_i)\frac{f[z_i, y_i, x_i]}{f[y_i, x_i]} \tag{11}$$