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Convergence and applications of some solutions of the confluent Heun equation

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ABSTRACT

We study the convergence of a group of solutions in series of confluent hypergeometric functions for the confluent Heun equation. These solutions are expansions in two-sided infinite series (summation from minus to plus infinity) which are interpreted as a modified version of expansions proposed by Leaver (1986). We show that the two-sided solutions yield two nonequivalent groups of one-sided series solutions (summation from zero to plus infinity). In the second place, we find that one-sided solutions of one of these groups can be used to solve an equation which describes a time-dependent two-level system of Quantum Optics. For this problem, in addition to finite-series solutions, we obtain infinite-series wavefunctions which are convergent and bounded for any value of the time t, and vanish when t goes to infinity.

1. Introductory remarks

Recently we have found a new solution for the ordinary spheroidal wave equation [1]. It is given by an expansion in series of irregular confluent hypergeometric functions $\Psi(a, c; y)$ with parameters a and b, and argument y. That expansion is a one-sided series in the sense that the summation, indicated by the index n, runs from zero to plus infinity ($n \geq 0$). From that solution we have inferred, without affording details, a group of solutions for the confluent Heun equation (CHE), given also by expansions in series of Ψ. Now we provide a detailed derivation of the solutions and their convergence.

Actually, we generalize the previous study by considering a group constituted by sets of three solutions: one in series of regular confluent hypergeometric functions $\Phi(a, c; y)$ and two in series of irregular functions $\Psi(a_i, c_i; y)$ (i = 1,2). From an initial set of solutions, other sets follow systematically by means of substitutions of variables which preserve the form of the CHE. The inclusion of the functions $\Phi(a, c; y)$ gives solutions valid near the origin $y = 0$. As a further generalization, we introduce an arbitrary parameter into the solutions and get two-sided series expansions ($-\infty < n < \infty$) which are necessary to treat problems where there is no free parameter in the CHE. We will find that these two-sided series can be interpreted as a modified version of Leaver's solutions [2].

We start with the two-sided series solutions and, from these, obtain the one-sided solutions as follows.

- We derive the group of two-sided series solutions and study the convergence of the solutions; then, we establish relations between these solutions and solutions found by Leaver in 1986 [2].
- We truncate the aforementioned two-sided series in order to find two different groups of one-sided series solutions.

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- Finally, we apply one-sided solutions to a two-state system which represents the interaction of an atom with a pulse of Lorentzian shape [3].

In this section we write the CHE, present the tests for determining the series convergence and explain the procedure for truncating the two-sided series; then, we outline the structure of the paper. We use the CHE in the form [2]

$$z(z-z_0)\frac{d^2U}{dz^2}+(B_1+B_2z)\frac{dU}{dz}+[B_3-2\omega\eta(z-z_0)+\omega^2z(z-z_0)]U=0,\quad \omega\neq 0, \tag{1}$$

where z_0, B_i, η and ω are constants. If $z_0\neq 0$, then $z=0$ and $z=z_0$ are regular singular points with indicial exponents $(0,1+B_1/z_0)$ and $(0,1-B_2-B_1/z_0)$, respectively. The point $z=\infty$ is an irregular singularity where the solutions behave as [2,4]

$$U(z)\sim e^{\pm i\omega z}\,z^{\mp i\eta-\frac{B_2}{2}},\quad z\to\infty. \tag{2}$$

The CHE is also called generalized spheroidal wave equation [2,5–7] but the last terminology sometimes is attached to a particular case of the CHE [8,9]. A limit of Eq. (1), called reduced CHE, is introduced in Section 5.

We deal with solutions whose series coefficients satisfy three-term recurrence relations, and use the theory concerning the three-term relations [10,11] to study the convergence. The general form of the two-sided series solutions is

$$U(z)=\sum_{n=-\infty}^{\infty}b_n^{\mu}\,h_n^{\mu}(z), \tag{3}$$

where the coefficients b_n^{μ} and the functions $h_n^{\mu}(z)$ depend on the parameters of the CHE and on a (characteristic) parameter μ to be determined – in principle, each set of solutions presents a different parameter denoted as μ_i. By omitting the parameter μ, the form of the recurrence relations for b_n^{μ} is

$$\alpha_n\,b_{n+1}+\beta_n\,b_n+\gamma_n\,b_{n-1}=0,\quad -\infty<n<\infty, \tag{4}$$

where α_n, β_n and γ_n depend on the parameters of the CHE and on μ. Equivalently,

$$\begin{bmatrix} \ddots & \ddots & \ddots & & & \\ & \gamma_n & \beta_n & \alpha_n & & \\ & & \gamma_{n+1} & \beta_{n+1} & \alpha_{n+1} & \\ & & & \gamma_{n+2} & \beta_{n+2} & \alpha_{n+2} \\ & & & & \ddots & \ddots & \ddots \end{bmatrix}\begin{bmatrix}\vdots\\ b_{n-1}\\ b_n\\ b_{n+1}\\ \vdots\end{bmatrix}=\mathbf{0},\quad -\infty<n<\infty, \tag{5}$$

where $\mathbf{0}$ denotes the null column vector. This system of homogeneous equations has non-trivial solutions only if the determinant of the above tridiagonal matrix vanishes. This condition affords the possible values for μ if there is no free constant in the CHE. If there is an arbitrary constant (and only in this case), we can attribute any convenient value for μ, whereas the condition on the determinant permits to find values for the arbitrary constant of the CHE.

The convergence of the two-sided series comes from the ratios

$$L_1(z)=\left|\frac{b_{n+1}h_{n+1}(z)}{b_nh_n(z)}\right| \text{ when } n\to\infty,\quad\text{and}\quad L_2(z)=\left|\frac{b_{n-1}h_{n-1}(z)}{b_nh_n(z)}\right| \text{ when } n\to-\infty. \tag{6}$$

By the D'Alembert test the series converges in the intersection of the regions where $L_1<1$ and $L_2<1$, and diverges otherwise excepting the inconclusive case $L_1=L_2=1$. Sometimes it is possible to decide about the convergence when $L_1=L_2=1$ by means of the Raabe test [12,13]. In effect, if for some value of z,

$$L_1(z)=1+\frac{A}{n}+O\left(\frac{1}{n^2}\right),\quad L_2(z)=1+\frac{B}{|n|}+O\left(\frac{1}{n^2}\right):\quad |n|\to\infty, \tag{7}$$

where A and B are constants, then the Raabe test states that the series converges if $A<-1$ and $B<-1$, and diverges otherwise (for $A=B=-1$ the test is inconclusive). In general, the limits of L_1 and L_2 afford different regions of convergence. Since the one-sided infinite series require only one of these limits, their domain of convergence may be larger than the domain of the corresponding two-sided series.

In fact, from two-sided series we will obtain two groups of one-sided series by taking into account that: (i) the series begins at $n=N+1$ if $\alpha_{n=N}=0$, where N is an integer (see page 171 of [14]); (ii) the series terminates at $n=M$ if $\gamma_{n=M+1}=0$ (page 146 of [14]). We write this as

$$\alpha_{n=N}=0\quad\Rightarrow\quad\text{series with } n\geq N+1;\quad \gamma_{n=M+1}=0\quad\Rightarrow\quad\text{series with } n\leq M. \tag{8}$$

So, to find the two groups of one-sided solutions, we suppose that there is a free constant in the CHE and choose the parameter μ such that

$$\begin{aligned}&\alpha_{-1}=0\ \Rightarrow\ \text{first group: series with } n\geq 0;\\ &\gamma_1=0\ \Rightarrow\ \text{second group: series with } n\leq 0.\end{aligned} \tag{9}$$