Applied Mathematics and Computation 284 (2016) 286–307

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Analysis of two active set type methods to solve unilateral contact problems

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ARTICLE INFO

MSC:
74M15
74G15
74B05
74B20
74S30
49M15
90C53
90C25
65K15

Keywords:
Unilateral constraint
Hertzian contact
Hyperelasticity
Projection iterative method
Primal dual active set
Augmented Lagrangian

ABSTRACT

In this work two active set type methods are considered in order to solve a mathematical problem which describes the frictionless contact between a deformable body and a perfectly rigid obstacle, the so-called Signorini Problem. These methods are the primal dual active set method and the projection iterative method. Our aim, here, is to analyze these two active set type methods and to carry out a comparison with the well-known augmented Lagrangian method by considering two representative contact problems in the case of large and small deformation. After presenting the mechanical formulation in the hyperelasticity framework, we establish weak formulations of the problem and the existence result of the weak solution is recalled. Then, we give the finite element approximation of the problem and a description of the numerical methods is presented. The main result of this work is to provide a convergence result for the projection iterative method. Finally, we present numerical simulations which illustrate the behavior of the solution and allow the comparison of the numerical methods.

1. Introduction

Various contact boundary conditions have been used to model contact phenomena, both in engineering and mathematics in the literature; see for instance [10,11,24,25,27] and the references therein for the mathematical analysis and [12,17,20,21,28] for the numerical analysis. One of the most popular remains the so-called Signorini condition, introduced in [26], which describes the contact with a perfectly rigid foundation. Expressed in terms of unilateral constraints for the displacement field, this condition leads to a non-linear and nonsmooth mathematical problem. To overcome these difficulties, several methods were used. For instance, Alart and Curnier presented in [1] an augmented Lagrangian formulation combined with a Generalized Newton method to solve non differentiable but continuous equations arising from frictional contact problems. Over the last few years, several other methods emerged. Amongst them, the active set strategies have been relatively popular for the last decade or so. The aim of this kind of method is to find the correct subset $\mathcal{A}$ of all nodes that are currently in contact with the perfectly rigid obstacle; those nodes are called active while the others are inactive, see [16,18] and references therein for more details. Also, one of the most interesting aspects of such a method is that it does not require the use of the Lagrange multipliers and, therefore, could facilitate the implementation of the algorithm and improve the condition number of the system. Indeed, since the boundary conditions on the contact boundary for the active

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http://dx.doi.org/10.1016/j.amc.2016.03.012
0096-3003/

nodes are directly enforced, we only get to solve a series of problems with simple boundary conditions, such as Dirichlet, Neumann or Robin boundary condition, based on the active set type method used. The purpose of the present work is to study and compare two active set type methods, the primal dual active set method and the projection iterative method, in the case of large and small deformation theories. The first one is the primal dual active set method used in various works such as [15,16] and [18] and the reference therein. In [16], the Authors study a class of semismooth Newton methods for quadratic minimization problems to non negativity constraints in which global and local non-linear convergence results of the resulting primal dual active set strategy were established under strong assumptions on the matrix of the linearized systems. The proof is based on the M-matrix properties of the discrete operator; while such a statement is true for the Laplace operator it is not for the discrete elasticity operator, as the authors admitted. Furthermore, in that work, only linear elasticity problems with unilateral boundary constraints are considered. The work in [18] is devoted to provide an inexact primal dual active set approach to solve non-linear multibody contact problems for linear elasticity. The non-linearity of such problems arise only from the non penetration conditions for the involved bodies in contact. The second active set type method was used for instance in [29] to solve the so-called Signorini problem for the Laplace equations and to obtain a convergence analysis of the method. This active set method is based on a fixed point equation and can be formulated as a projection iterative algorithm. The projection iterative method consists to solve a sequence of Dirichlet or Robin boundary conditions according to a contact criteria in order to find the correct sets of active and inactive contact nodes. Our aim in this work is twofold. First, we present and analyze the active set type methods compared to the well-known augmented Lagrangian approach to solve a Signorini contact problem for hyperelasticity. Furthermore, we extend the convergence result obtained in [29] for the projection iterative method to a Signorini problem in the case of linear elasticity. Next we provide a numerical comparison of the different methods by considering simulations on two-dimensional test problems: one in the small deformation framework, the Hertz contact problem, and one in the large deformation framework, the contact between a hyperelastic ring and a rigid foundation under strong compressions. In particular, we analyze and compare the numerical convergence of the different methods with respect to several parameters such as the number of degrees of freedom or the number of iterations. By doing so, we also illustrate the behavior of the solution related to the contact condition.

The rest of the paper is structured as follows. In Section 2 we describe the contact conditions and introduce the mechanical problem in the large deformation framework. Then we introduce the notation and some preliminary material, list the assumptions on the data and state the variational formulation of the problem in the hyperelasticity framework. In Section 3, we provide the finite element approximation of the variational formulation. After reformulating the problem into a minimization one, an augmented Lagrangian method is recalled to treat the unilateral constraints. In Section 4, we describe the two active set type methods used to solve unilateral contact problems in hyperelasticity: the primal dual active set method and the projection iterative method. The Section 5 is devoted to the convergence analysis of the projection iterative method to the solution of the Signorini Problem, in the small deformation hypothesis. After that, in Section 6 we present several numerical simulations to illustrate and compare the behavior of the two active set type methods. Finally, in Section 7, we conclude by recalling the results obtained and discussing about the conceivable works in the continuation of this one.

2. Formulations of the contact problem

The purpose of this section is to present both the mechanical problem and the variational formulation in the framework of hyperelasticity.

2.1. Hyperelastic contact model

The physical setting of the mechanical problem is as follows. A hyperelastic body occupies a bounded domain $\Omega \subset \mathbb{R}^d$ $(d = 1, 2, 3)$ with a Lipschitz continuous boundary Γ, divided into three disjoint measurable parts Γ_1, Γ_2 and Γ_3 such that $meas(\Gamma_1) > 0$. The notation $\boldsymbol{x} = (x_i)$ for a typical point in $\Omega \cup \Gamma$ is used and we denote by $\boldsymbol{\nu} = (\nu_i)$ the outward unit normal at Γ. Here and below the indices i, j, k, l run between 1 and d and, unless stated otherwise, the summation convention over repeated indices is used. We denote by $\mathbb{M}^d$ the space of second order tensors on $\mathbb{R}^d$ or, equivalently, the space of square matrices of order d. The inner product and norm on $\mathbb{R}^d$ and $\mathbb{M}^d$ are defined by

$$\boldsymbol{u} \cdot \boldsymbol{v} = u_i v_i\ , \qquad \|\boldsymbol{v}\| = (\boldsymbol{v} \cdot \boldsymbol{v})^{\frac{1}{2}} \qquad \forall\, \boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^d,$$
$$\boldsymbol{\Pi} \cdot \boldsymbol{\tau} = \Pi_{ij}\tau_{ij}\ , \qquad \|\boldsymbol{\tau}\| = (\boldsymbol{\tau} \cdot \boldsymbol{\tau})^{\frac{1}{2}} \qquad \forall\, \boldsymbol{\Pi}, \boldsymbol{\tau} \in \mathbb{M}^d.$$

We use the notation $\boldsymbol{u}$ and $\boldsymbol{\Pi}$ for the displacement field and the first Piola–Kirchoff stress tensor, respectively. Also, we denote by u_ν and $\boldsymbol{u}_\tau$ the normal stress and tangential components of $\boldsymbol{u}$ on Γ given by $v_\nu = \boldsymbol{v} \cdot \boldsymbol{\nu}$, $\boldsymbol{v}_\tau = \boldsymbol{v} - v_\nu \boldsymbol{\nu}$. Finally, Π_ν and $\boldsymbol{\Pi}_\tau$ will represent the normal and the tangential stress on Γ, defined by $\Pi_\nu = (\boldsymbol{\Pi}\boldsymbol{\nu}) \cdot \boldsymbol{\nu}$ and $\boldsymbol{\Pi}_\tau = \boldsymbol{\Pi}\boldsymbol{\nu} - \Pi_\nu \boldsymbol{\nu}$. Furthermore, an index that follows a comma represents the partial derivative with respect to the corresponding component of the spatial variable $\boldsymbol{x}$, e.g. $u_{i,j} = \partial u_i / \partial x_j$. Moreover, we recall that the divergence operator is defined by the equality $\mathrm{Div}\, \boldsymbol{\Pi} = (\Pi_{ij,j})$.

In the problems studied below, the material's behavior is described with a hyperelastic constitutive law. We recall that hyperelastic constitutive laws are characterized by the first Piola–Kirchhoff tensor $\boldsymbol{\Pi}$ which derives from an internal hyperelastic energy density $W : \Omega \times \mathbb{M}^d_+ \to \mathbb{R}$, $\boldsymbol{\Pi} = \frac{\partial}{\partial \mathbf{F}} W(\boldsymbol{x}, \mathbf{F}) = \partial_{\mathbf{F}} W(\boldsymbol{x}, \mathbf{F})$, for all $\boldsymbol{x} \in \Omega$ and $\mathbf{F} \in \mathbb{M}^d$. Here $\mathbf{F}$ is the deformation