Applied Mathematics and Computation 281 (2016) 121–129

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

A novel computational hybrid approach in solving Hankel transform

Nagma Irfan*, A.H. Siddiqi

Department of Mathematics, School of Basic Sciences and Research, Sharda University, Greater Noida-201306, India

ARTICLE INFO

MSC:
44A15
65R10

Keywords:
Hankel transforms
Bessel functions
Hybrid of Block-pulse functions and Taylor polynomials
Noise terms

ABSTRACT

In this paper, we use a combination of Taylor and block-pulse functions on the interval [0, 1], that is called Hybrid Functions to estimate fast and stable solution of Hankel transform. First hybrid of Block-Pulse and Taylor polynomial basis is obtained and orthonormalized using Gram–Schmidt process which are used as basis to expand a part of the integrand,$rf(r)$ appearing in the Hankel transform integral. Thus transforming the integral into a Fourier–Bessel series. Truncating the series, an efficient stable algorithm is obtained for the numerical evaluation of the Hankel transforms of order$\nu > -1$. The novelty of our method is that we give error analysis and stability of the hybrid algorithm and corroborate our theoretical findings by various numerical experiments for the first time. The solutions obtained by projected method indicate that the approach is easy to implement and computationally very attractive.

1. Introduction

In this paper, we use a combination of Taylor & Block-Pulse functions on the interval [0, 1], that is called hybrid functions, to estimate numerical solution of Hankel Transform. In Recent years, many different basic functions have used to estimate Hankel transform. In our hybrid method we use simple basis, Hybrid of block pulse and Taylor polynomials which are used in solving many engineering problems [1–4].

1.1. Hankel transform: Definition

Several definitions of the Hankel transform appear in the literature. In this paper, we use the definition of the νth order Hankel transform as defined by Piessens [5] to define the Hankel transform as

$$F_\nu(p) \equiv \chi_\nu\{f(r)\} \equiv \int_0^\infty rf(r)J_\nu(pr)dr \tag{1}$$

Here, ν may be an arbitrary real or complex number. However, an integral transform needs to be invertible in order to be useful and this restricts the allowable values of ν. If ν is real and $\nu > -1/2$, and under suitable conditions of integrability of the function, the transform is self-reciprocating and the inversion formula is given by

$$f(r) = \chi_\nu^{-1}\{F_\nu(p)\} \equiv \int_0^\infty pF_\nu(p)J_\nu(pr)dp. \tag{2}$$

* Corresponding author: Tel.: +91 9811667135.
E-mail address: nagmairfanmath@gmail.com, nagmamath@gmail.com (N. Irfan).

http://dx.doi.org/10.1016/j.amc.2016.01.028

where J_ν is the νth order Bessel function of first kind. Thus, the Hankel transforms takes a function $f(r)$ in the spatial r domain and transforms it to a function in the frequency $F_\nu(p)$ domain. This relationship is denoted symbolically as f(r) ⇔F (ρ). The oscillatory nature of J_ν and infinite range of the integral renders its numerical evaluation difficult. Analytical evaluations are rare and their numerical computations are difficult because of the oscillatory behavior of the Bessel function and the infinite length of the interval.

To overcome these hitches, various different techniques are available in the literature. Numerous papers have been carved to the numerical evaluation of the Hankel Transform.

In general there are two methods which are popular for effective calculation in this area these are as follows:

1) Fast Hankel Transform:

Here, by substitution and scaling, the problem is transformed in the space of the logarithmic co-ordinates and the fast Fourier transform in that space. In this method smoothing of the function in log space is required. This method is proposed by Siegman in [6].

2) Filon quadrature philosophy:

This is one of popular method for effective calculation in this area. In Filon quadrature philosophy, the integrand is separated into the product of an (assumed) slowly varying component and a rapidly oscillating component [7]. In the context of the Hankel transform, the former is $rf(r)$ and the latter is $J_\nu(pr)$. This method works quite well for computing $F_0(p)$, for $p \geq 1$, but the calculation of inverse Hankel transform is more difficult, as $F_0(p)$ is no longer a smooth function but a rapidly oscillating one. Moreover the error is appreciable between$0 < p < 1$.

Numerical estimation of HT is ubiquitous in the mathematical treatment of physical problems involving cylindrical symmetry, optics, Seismology, Electromagnetism and astronomy etc. Many different types of algorithms and software have been developed to evaluate Hankel Transform [8–16] and applications are used in scientific problems [17–19].

The properties of Hybrid of block pulse and Taylor polynomials are presented in this method and are utilized to reduce the computation of Hankel transform as comparable to previous algorithms. The lack of error estimation in the papers cited above inspired us for present work. The test functions with known analytic HT are used, to illustrate the stability and efficiency of the proposed algorithm. First Orthonormal hybrid functions are obtained and used as basis for approximation of $rf(r)$ and an efficient and stable algorithm is given. Further corroborated by the numerical experiments performed on 3 test functions of different orders. The test functions with known analytic HT are used with random noise term $\varepsilon\,\theta_i$ added to the data function $rf(r)$, where θ_i is a uniform random variable with values in $[-1, 1]$, to illustrate the stability and efficiency of the proposed algorithm.

2. Properties of Hybrid functions

2.1. Hybrid functions of Block-pulse and Taylor polynomials

Definition. A set of Block-Pulse functions $\varphi_i(\lambda)$; $i = 1, 2, \ldots, m$ on the interval [0,1] is defined as follows:

$$\varphi_i(\lambda) = \begin{cases} 1 & \dfrac{i-1}{m} \leq \lambda < \dfrac{i}{m} \\ 0 & \text{otherwise.} \end{cases}$$

The Block-Pulse functions on [0,1] are disjoint, that is for $i=1, 2, \ldots, m$, $j=1, 2, \ldots, m$ we have $\varphi_i(\lambda)\varphi_j(\lambda) = \delta_{ij}\varphi_i(\lambda)$, also these functions have the property of orthogonality on [0,1].

Consider the Taylor polynomials $T_m(t)=t^m$; $m=0, 1, 2, \ldots$ on the interval $[0, 1]$.

Hybrid functions of block pulse and Taylor polynomials $\chi_{nm}(t)$, $n = 1, 2, \ldots, N$, $m = 0, 1, \ldots, M-1$, are defined as

$$\chi_{nm}(t) = \begin{cases} T_m(Nt - (n-1) & \dfrac{n-1}{N} \leq t < \dfrac{n}{N}, \text{[1]} \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

3. Orthonormal basis functions

Using Gram–Schmidt orthogonalization process on $\chi_{nm,}$ first we obtain a class of orthogonal polynomials from hybrid functions of Block pulse and Taylor polynomials denoted by $\xi_{nm}(t)$ and defined for n=3 and m=4 by as