Applied Mathematics and Computation 281 (2016) 152–171

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

# Constructing transient birth–death processes by means of suitable transformations

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ARTICLE INFO

*MSC:*
60J80
60J28
65C50

*Keywords:*
Conditional probabilities
First-passage time
Catastrophes
$\nu$-similarity
Immigration-birth–death process
Linear immigration-death process

ABSTRACT

For a birth–death process $N(t)$ with a reflecting state at 0 we propose a method able to construct a new birth–death process $M(t)$ defined on the same state-space. The birth and death rates of $M(t)$ depend on the rates of $N(t)$ and on the probability law of the process $N(t)$ evaluated at an exponentially distributed random time. Under a suitable assumption we obtain the conditional probabilities, the mean of the process, and the Laplace transforms of the downward first-passage-time densities of $M(t)$. We also discuss the connection between the proposed method and the notion of $\nu$-similarity, as well as a relation between the distribution of $M(t)$ and the steady-state probabilities of $N(t)$ subject to catastrophes governed by a Poisson process. We investigate new processes constructed from (i) a birth–death process with constant rates, and (ii) a linear immigration-death process. Various numerical computations are performed to illustrate the obtained results.

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## 1. Introduction

Birth–death processes and related stochastic models are relevant in several fields, such as population dynamics, evolutionary genomics, ecology, queueing theory and inventory, among others (see, for instance, Ricciardi [1], Renshaw [2], Crawford and Suchard [3], Di Crescenzo et al. [4], Dharmaraja et al. [5]). Moreover, birth–death processes have recently been considered prominently in spatial evolutionary games for the analysis of cooperation and evolution in binary birth–death dynamics and for the expansion of cooperation by means of self-organized growth (see Szolnoki et al. [6]).

Many applications often demand for the determination of the probability laws of such processes, which is not an easy task. In fact various techniques have been developed in the past decades aiming to obtain the transition probabilities of birth–death processes. Some methods are based on transforms, such as generating functions and Laplace transforms, spectral decompositions (see van Doorn [7,8]) or continued fractions. Attention has been given also to the use of suitable transformations (see Lenin and Parthasarathy [9] and Lenin et al. [10]) and direct methods (see Parthasarathy [11]).

Along this line, in the present paper we propose a method able to determine closed-form transition probabilities of certain time-homogeneous birth–death processes with 0 reflecting state. This method is based on a transformation between two birth–death processes such that the birth and death rates of the new process depends on those of the former process and on the survival probability of a compound random variable, say $Z$. Specifically, $Z$ describes the former birth–death process evaluated at an exponentially distributed random time $T$. We point out that the proposed method leads to transient

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http://dx.doi.org/10.1016/j.amc.2016.01.058
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birth–death processes, which are appropriate for describing population subject to rapid growth (such as unbounded bacterial growth). For these processes we determine various quantities of interest via computationally effective procedures, such as the conditional probabilities, the mean and Laplace transforms of some first-passage-time densities.

We point out that a further object of the paper is to illustrate a suitable connection between the obtained processes and the former processes subject to total catastrophes. We remark that some results on stochastic processes subject to catastrophes have been shown recently in Cairns and Pollett [12], van Doorn and Zeifman [13], Pollett et al. [14], Di Crescenzo et al. [15], Dimou and Economou [16], Zeifman et al. [17].

It is worth noting that the potential applicability of the proposed theory can be extended also to the physics of social systems, as reviewed in Castellano et al. [18], as well as to statistical mechanics of evolutionary and coevolutionary games, as reviewed recently in Perc and Grigolini [19].

We recall that the problem of determining the probability distribution of Markov chains is often unfeasible and thus one is forced to resort to suitable approximations, such as those based on convergences of truncated processes (see, e.g. Tweedie [20]). Moreover, the criteria grounded on truncated birth–death processes are often constrained by certain conditions, such as monotonicity or boundedness of transition rates (see Zeifman et al. [21,22]. The procedure proposed in this paper allows to obtain exact distributions rather than approximate ones. Furthermore, the approximation based on truncated birth–death processes in our case is not necessarily successful, since the transition rates can be unbounded (see Section 6.2).

This is the plan of the paper. In Section 2 we present the method, based on the transformation between birth–death processes, both having state-space $\mathbb{N}_0$, with 0 a reflecting state. Starting from a birth–death process $N(t)$, we define a new birth–death process $M(t)$, whose birth and death rates depend on those of $N(t)$ and on the probability law of the process $N(t)$ evaluated at an exponentially distributed random time $T$ with mean $\xi^{-1} > 0$. We obtain the conditional probabilities and the mean of $M(t)$ in closed form, and investigate the special case when $\xi \downarrow 0$. A remark on the case dealing with general birth processes is also given.

Section 3 is centered on the connection between the proposed method and the notion of $\nu$-similarity, which is shown to hold for a special family of immigration-birth–death processes.

In Section 4 we show a connection between the distribution of $M(t)$ and the steady-state probability of $N(t)$ subject to catastrophes occurring according to a Poisson process with rate $\xi$.

Section 5 deals with Laplace transforms and first-passage time. We determine the Laplace transforms of the conditional probabilities of $M(t)$. Such functions are used to obtain the Laplace transform of the downward first-passage-time densities. Some results on the asymptotic behavior of the rates of $M(t)$ are also shown.

In Section 6 we analyze some special cases. We apply the proposed method to (i) a birth–death process with constant rates, and (ii) a linear immigration-death process. Various numerical computations are performed by means of MATHEMATICA® to illustrate the obtained results and to elucidate the role of the parameters.

## 2. Main results

Let $\{N(t),\ t \geq 0\}$ be a continuous-time birth–death process with state space $\mathbb{N}_0 = \{0, 1, \dots\}$, 0 being a reflecting state. Assume that the birth rates $\{\lambda_n,\ n \in \mathbb{N}_0\}$ and the death rates $\{\mu_n,\ n \in \mathbb{N}\}$ are strictly positive, so that the process $N(t)$ is irreducible. As usual, we denote by

$$\pi_0 = 1, \quad \pi_n = \frac{\lambda_0 \lambda_1 \dots \lambda_{n-1}}{\mu_1 \mu_2 \dots \mu_n}, \quad n \in \mathbb{N} \tag{1}$$

the potential coefficients of $N(t)$ (cf. Keilson [23]). We remark that

$$\lambda_n \pi_n = \mu_{n+1} \pi_{n+1} \quad \text{for all } n \in \mathbb{N}_0. \tag{2}$$

Let us introduce the following notation (cf. Callaert and Keilson [24], Kijima [25]):

$$A = \sum_{n=0}^{+\infty} \frac{1}{\lambda_n \pi_n}, \quad B = \sum_{n=0}^{+\infty} \pi_n,$$
$$C = \sum_{n=0}^{+\infty} \frac{1}{\lambda_n \pi_n} \sum_{i=0}^{n} \pi_i, \quad D = \sum_{n=0}^{+\infty} \frac{1}{\lambda_n \pi_n} \sum_{i=n+1}^{\infty} \pi_i. \tag{3}$$

We recall that the process $N(t)$ is recurrent if $A = +\infty$ and is transient if $A$ is finite. If $N(t)$ is recurrent, then it is positive recurrent if $B$ is finite and null recurrent if $B$ diverges. Moreover, the boundary at infinity of $N(t)$ is regular if $C$ and $D$ are finite, exit if $C < +\infty$ and $D = +\infty$, entrance if $C = +\infty$ and $D < +\infty$ and natural if $C = D = +\infty$. If the series $C$ diverges, then the birth–death process $N(t)$ is nonexplosive. This classification is summarized in Anderson [26], page 262. Other general aspects are treated in Feller [27].

Hereafter, we assume that the boundary at infinity of $N(t)$ is natural, so that the process is nonexplosive and all the states are transient or recurrent. We denote the transition probabilities of $N(t)$ by

$$p_{j,n}(t) = P\{N(t) = n \mid N(0) = j\} \quad (j, n \in \mathbb{N}_0). \tag{4}$$

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