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Impulsive fractional q-integro-difference equations with separated boundary conditions

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ABSTRACT

In this paper, we discuss the existence of solutions for impulsive fractional q-integro-difference equations with separated boundary conditions. Existence results are proved via fixed point theorems due to Krasnoselskii and O'Regan, while the uniqueness of solutions is accomplished by means of Banach's contraction mapping principle. Examples illustrating the obtained results are also presented.

1. Introduction

Quantum difference operators (q-difference operators) have extensive applications in diverse disciplines such as orthogonal polynomials, basic hyper-geometric functions, combinatorics, the calculus of variations, mechanics and the theory of relativity. For a detailed description of such operators, we refer a text Kac and Cheung [1].

In classical quantum calculus (q-calculus), the q-derivative was first formulated by Jackson [2] in 1910 as

$$D_q x(t) = \frac{x(t) - x(qt)}{(1-q)t}, \quad 0 < q < 1, \ t \in (0, \infty). \tag{1.1}$$

The above definition does not remain valid for impulse points t_k, $k \in \mathbb{Z}$, such that $t_k \in (qt, t)$. On the other hand, this situation does not arise for impulsive equations on q-time scales as the domains consist of isolated points covering the case of consecutive points of t and qt with $t_k \notin (qt, t)$. Due to this reason, the subject of impulsive quantum difference equations on dense domains could not be studied. In [3], the authors modified the classical quantum calculus for obtaining the first and second order impulsive quantum difference equations on a dense domain $[0, T] \subset \mathbb{R}$ through the introduction of a new q-shifting operator defined by ${}_a\Phi_q(m) = qm + (1-q)a$, $m, a \in \mathbb{R}$. If $a < m$, then $a < {}_a\Phi_q(m) < m$. Let t_k, t_{k+1} be consecutive impulse points and $[t_k, t_{k+1}]$ be a dense subset of $\mathbb{R}$. For $t \in [t_k, t_{k+1}]$, we have ${}_{t_k}\Phi_q(t) \in (t_k, t_{k+1})$. The main idea was to

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apply quantum calculus only on a sub-interval (t_k, t_{k+1}) and then combine all intervals through impulsive conditions. In [4], the authors used the q-shifting operator to develop the new concepts of fractional quantum calculus such as the Riemann–Liouville fractional derivative and integral and their properties. They also formulated the existence and uniqueness results for some classes of first and second orders impulsive fractional q-difference equations.

Impulsive differential equations serve as basic models to study the dynamics of processes that are subject to sudden changes in their states. Recent development in this field has been motivated by many applied problems arising in control theory, population dynamics and medicines. For some recent works on the theory of impulsive differential equations, we refer the reader to the monographs [5–7].

Recently, in [8], the authors applied the concepts of quantum calculus developed in [3] to study a boundary value problem of ordinary impulsive q_k-integro-difference equations of the form:

$$\begin{cases} D_{q_k}^2 x(t) = f(t, x(t), (S_{q_k}x)(t)), \quad t \in J := [0,T], \ t \neq t_k, \\ \Delta x(t_k) = I_k(x(t_k)), \quad k = 1,2,\ldots,m, \\ D_{q_k}x(t_k^+) - D_{q_{k-1}}x(t_k) = I_k^*(x(t_k)), \quad k = 1,2,\ldots,m, \\ x(0) + D_{q_0}x(0) = 0, \quad x(T) + D_{q_m}x(T) = 0, \end{cases} \tag{1.2}$$

where $0 = t_0 < t_1 < t_2 < \cdots < t_k < \cdots < t_m < t_{m+1} = T$, $f : J \times \mathbb{R}^2 \to \mathbb{R}$, $(S_{q_k}x)(t) = \int_{t_k}^t \phi(t,s)x(s)d_{q_k}s, t \in (t_k, t_{k+1}]$, $k = 0,1,2,\ldots,m$, ϕ: $J \times J \to [0, \infty)$ is a continuous function, $I_k, I_k^* \in C(\mathbb{R},\mathbb{R})$, $\Delta x(t_k) = x(t_k^+) - x(t_k)$ for $k = 1,2,\ldots,m$, $x(t_k^+) = \lim_{h\to 0} x(t_k + h)$ and $0 < q_k < 1$ for $k = 0,1,2,\ldots,m$. The second order q_k-difference appeared in (1.2) is defined by $D_{q_k}^2 x = D_{q_k}(D_{q_k}x)$, where the first order q_k-difference operator is

$$D_{q_k}x(t) = \frac{x(t) - x({}_{t_k}\Phi_{q_k}(t))}{(1-q_k)(t-t_k)}, t \neq t_k, \quad D_{q_k}x(t_k) = \lim_{t\to t_k} D_{q_k}x(t). \tag{1.3}$$

Some existence and uniqueness results for problem (1.2) were proved by using a variety of fixed point theorems. More recently, the authors discussed the existence of solutions for Caputo–Hadamard type fractional impulsive hybrid systems with nonlinear fractional integral conditions in [9].

The aim of this paper is to present a new definition of Caputo type quantum difference operator and investigate the existence criteria for the solutions of an impulsive fractional q-integro-difference equation involving this operator supplemented with separated boundary conditions given by

$$\begin{cases} {}_{t_k}^{c}D_{q_k}^{\alpha_k} x(t) = f(t, x(t), {}_{t_k}I_{q_k}^{\beta_k}x(t)), \quad t \in J_k \subseteq [0,T], \ t \neq t_k, \\ \Delta x(t_k) = x(t_k^+) - x(t_k) = \varphi_k(x(t_k)), \quad k = 1,2,\ldots,m, \\ {}_{t_k}D_{q_k}x(t_k^+) - {}_{t_{k-1}}^{c}D_{q_{k-1}}^{\gamma_{k-1}}x(t_k) = \varphi_k^*(x(t_k)), \quad k = 1,2,\ldots,m, \\ \lambda_1 x(0) + \lambda_2 \, {}_0D_{q_0}x(0) = 0, \qquad \xi_1 x(T) + \xi_2 \, {}_{t_m}^{c}D_{q_m}^{\gamma_m}x(T) = 0, \end{cases} \tag{1.4}$$

where $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = T$, ${}_{t_k}^{c}D_{q_k}^{\phi_k}$ denotes the Caputo q_k-fractional derivative of order $\phi_k \in \{\alpha_k, \gamma_k\}$ on J_k, $1 < \alpha_k \le 2$, $0 < \gamma_k \le 1$, $0 < q_k < 1$, $J_0 = [0,t_1]$, $J_k = (t_k, t_{k+1}]$, $k = 0,1,\ldots,m$, $J = [0,T]$, $f \in C(J \times \mathbb{R}^2, \mathbb{R})$, $\varphi_k, \varphi_k^* \in C(\mathbb{R},\mathbb{R})$, $k = 1,2,\ldots,m$ and ${}_{t_k}I_{q_k}^{\beta_k}$ denotes the Riemann–Liouville q_k-fractional integral of order $\beta_k > 0$ on J_k, $k = 0,1,2,\ldots,m$. The key tools to study the given problem are fixed point results due to Krasnoselskii and O'Regan which require the segregation of an operator into a sum of two operators. Some new notations of quantum constants are introduced in (2.5) and (2.6) to facilitate the process of computing.

In recent years, the topic of q-calculus has attracted the attention of several researchers and a variety of new results on q-difference and fractional q-difference equations can be found in a series of books [10–12] and papers [13–31], and the references cited therein. Some applications of q-calculus have appeared in [32–36], but these applications do not take into account the impulsive effects. The results obtained in this paper will be useful to extend the study of these applications with impulse conditions.

The paper is organized as follows. Section 2 contains the basic concepts of q-calculus and an auxiliary lemma that lays the foundation to convert problem (1.4) into an equivalent fixed point operator equation. In Section 3, we discuss the main results, while their illustration is presented in Section 4.

2. Preliminaries

This section is devoted to some basic concepts such as q-shifting operator, Riemann–Liouville fractional q-integral and q-difference on a given interval [4].

We define a q-shifting operator as

$${}_a\Phi_q(m) = qm + (1-q)a. \tag{2.1}$$

For any positive integer k, we have

$${}_a\Phi_q^k(m) = {}_a\Phi_q^{k-1}({}_a\Phi_q(m)) \quad \text{and} \quad {}_a\Phi_q^0(m) = m.$$