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On graded meshes for a two-parameter singularly perturbed problem

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ABSTRACT

A one-dimensional reaction–diffusion–convection problem is numerically solved by a finite element method on two layer-adapted meshes, Duran-type mesh and Duran–Shishkin-type mesh, both defined by recursive formulae. Robust error estimates in the energy norm are proved. Numerical results are given to illustrate the parameter–uniform convergence of numerical approximations.

1. Introduction

Singularly perturbed differential equations are frequently characterized by a small (perturbation) parameter multiplying the highest order derivative. Solutions of such equations have layers that can appear in different parts of the problem domain. This is also the case in the presence of more than one perturbation parameter, as in the following boundary value problem

$$\begin{aligned} &Lu(x) := -\varepsilon_1 u''(x) + \varepsilon_2 b(x)u'(x) + c(x)u = f(x), \quad x \in \Omega = (0,1), \\ &u(0) = 0, \quad u(1) = 0. \end{aligned} \tag{1}$$

Classical computations to such problems are often inadequate since they require a large number of mesh points in order to produce a satisfactory computed solution.

The main goal in the construction of numerical methods for singularly perturbed problems is to obtain their uniform convergence (robustness) with respect to all perturbation parameters. If u is the solution of a singularly perturbed problem, and u^N its numerical approximation obtained by a numerical method with N degrees of freedom, then the numerical method is uniformly convergent (robust) with respect to the perturbation parameters in the norm $\|\cdot\|$, if

$$\|u - u^N\| \le \vartheta(N) \ \text{ for } \ N \ge N_0,$$

with the function ϑ that is independent of perturbation parameters and satisfies

$$\lim_{N\to\infty} \vartheta(N) = 0.$$

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Here, we shall use layer-adapted meshes, constructed in a special way, in order to ensure uniform convergence of a finite element method for the problem (1) with

$$b(x) \geq b_0 > 0, \quad c(x) \geq c_0 > 0, \quad c(x) - \frac{1}{2}\varepsilon_2 b'(x) \geq \gamma > 0, \quad x \in \Omega, \tag{2}$$

where b, c, f are smooth functions on $\overline{\Omega} = [0, 1]$, b_0, c_0, γ are constants and $0 < \varepsilon_1, \varepsilon_2 \ll 1$ are small perturbation parameters. This kind of problem often arises in chemical reactor theory, biology, transport phenomena in chemistry, electrical networks, control and lubrication theory, [1,5,13,14].

When $\varepsilon_2 = 0$, the problem (1) and (2) is a reaction–diffusion differential equation whose solution has two boundary layers of the widths $\mathcal{O}(\varepsilon_1^{1/2}|\ln \varepsilon_1^{1/2}|)$. For $\varepsilon_2 = 1$, the problem (1) and (2) reduces to a convection–diffusion with a solution exhibiting a layer at $x = 1$ of the width $\mathcal{O}(\varepsilon_1|\ln \varepsilon_1|)$. In both single parameter cases, there is a large number of references devoted to their numerical study, see for instance books [10,17] and the cited literature. Two-parameter problems are less discussed. In [8,12,15], the authors investigate uniform convergence of finite difference methods, while results for finite elements can be found in [9,19,21] for the standard Shishkin mesh, and in [2,9] for Bakhvalov-type meshes.

In this paper, the singularly perturbed problem (1) and (2) is numerically solved using Galerkin finite element method on two recursively generated meshes, Duran–Shishkin [18], and Duran mesh [6]. As already emphasized in [6,7,18], these meshes represent an interesting alternative to standard layer-adapted meshes due to their simple construction and error analysis. Unlike the Bakhvalov-type mesh from [2], they do not require the use of a special quasi-interpolant. In Sections 4 and 5, we shall show that it is sufficient to apply the standard Lagrange interpolation in the error analysis of the Galerkin method. Compared to the standard Shishkin mesh, [10,17], the authors in [6,7] noticed advantages of the Duran mesh in some numerical experiments, while the analysis from [18] shows a weaker dependence on a logarithmic factor of the Duran–Shishkin mesh. There are also other variants of graded meshes, [6,7,16,18], and they all give rise to almost optimal convergence results since they are locally quasi-equidistant.

The paper is organized as follows. After the solution properties, in Section 2 we present a corresponding weak formulation and the Galerkin approach. In Section 3, the aforementioned layer–adapted meshes are modified to fit the solution layers of the two-parameter problem (1) and (2). Sections 4 and 5 contain our main results on uniform error estimates (Theorem 4 and Theorem 5). Based on interpolation and discretization errors, we prove a robustness of the method in an energy norm. In Section 6, numerical results that support our theoretical findings are presented.

Notation 1. Throughout the paper, C will denote a generic positive constant independent of the perturbation parameters ε_1, ε_2, the number of degrees of freedom and the mesh parameter h. For a set $D \subset \mathbb{R}$, a standard notation for Banach spaces $L^p(D)$, Sobolev spaces $W^{k,p}(D)$, $H^k(D) = W^{k,2}(D)$, norms $\|\cdot\|_{L^p(D)}$ and seminorms $|\cdot|_{H^k(D)}$ has been used.

2. Solution properties and weak formulation

In order to describe the layers we introduce characteristic equation for (1)

$$-\varepsilon_1 r^2(x) + \varepsilon_2 b(x) r(x) + c(x) = 0,$$

which has two real solutions $r_0(x) < 0$ and $r_1(x) > 0$. Let

$$\mu_0 = -\max_{x\in[0,1]} r_0(x), \qquad \mu_1 = \min_{x\in[0,1]} r_1(x),$$

namely

$$\mu_{0,1} = \min_{x\in[0,1]} \frac{\mp\varepsilon_2 b(x) + \sqrt{\varepsilon_2^2 b^2(x) + 4\varepsilon_1 c(x)}}{2\varepsilon_1}.$$

Different properties of μ_0 and μ_1 can be derived, while here we emphasize the following, [20],

$$1 \ll \mu_0 \leq \mu_1, \qquad \max\{\mu_0^{-1}, \varepsilon_1\mu_1\} \leq C(\varepsilon_2 + \varepsilon_1^{1/2}), \tag{3}$$

$$\varepsilon_2\mu_0 \leq b_0^{-1}\|c\|_{L^\infty(\Omega)}, \qquad \varepsilon_2(\varepsilon_1\mu_1)^{-1/2} \leq C\varepsilon_2^{1/2}. \tag{4}$$

The solution u of the starting problem has two boundary layers according to the following

Theorem 1 [10]**.** *Let $b, c, f \in C^q(\overline{\Omega})$ for some $q \geq 1$, and let $p, \kappa \in (0, 1)$ be arbitrary. If*

$$q\|b'\|_{L^\infty(\Omega)}\varepsilon_2 \leq \kappa(1-p),$$

then

$$|u^{(k)}(x)| \leq C(1 + \mu_0^k e^{-p\mu_0 x} + \mu_1^k e^{-p\mu_1(1-x)}), \quad x \in \Omega, \quad 0 \leq k \leq q.$$

Now, from [11] the solution u has the representation $u = S + E_0 + E_1$, where $LS = f$, $LE_0 = 0$, $LE_1 = 0$ and

$$|S^{(k)}(x)| \leq C, \quad |E_0^{(k)}(x)| \leq C\mu_0^k e^{-p\mu_0 x}, \quad |E_1^{(k)}(x)| \leq C\mu_1^k e^{-p\mu_1(1-x)}, \tag{5}$$

for $x \in \Omega$ and $0 \leq k \leq q$.