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Stability of Newton TVD Runge–Kutta scheme for one-dimensional Euler equations with adaptive mesh☆

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ABSTRACT

In this paper, we propose a moving mesh method with a Newton total variation diminishing (TVD) Runge–Kutta scheme for the Euler equations. Our scheme improves time discretization in the moving mesh algorithms. By analyzing the semi-discrete Euler equations with the discrete moving mesh equations as constraints, the stability of the Newton TVD Runge–Kutta scheme is proved. Thus, we can conclude that the proposed algorithm can generate a weak solution to the Euler equations. Finally, numerical examples are presented to verify the theoretical results and demonstrate the accuracy of the proposed scheme.

1. Introduction

We are concerned with numerical solutions of the one-dimensional Euler equations. The Euler equations are a system of hyperbolic conservation laws that have important applications in the physical sciences and engineering fields such as solid and fluid dynamics, combustion, etc. [1]. Many problems (i.e., fronts or shocks) are characterized by moving discontinuities. A major challenge in obtaining the numerical solutions of these problems is to capture the discontinuous solutions with sufficient accuracy while keeping the computational cost within acceptable limits.

Since the contact discontinuities of solutions for nonlinear hyperbolic conservation laws often exhibit a wide range of localized structures, mesh adaptation is an indispensable tool for obtaining efficient numerical solutions. There are three types of mesh adaptive methods: *h*-methods, *p*-methods, and *r*-methods [2–4]. The *r*-methods are also called moving mesh methods, which relocate mesh point positions while maintaining the total number of mesh points and the mesh connectivity. In this paper, we study the moving mesh methods in order to obtain the numerical solution of a gas dynamics system. Some success has been achieved in solving hyperbolic problems on adaptive spatial meshes. Harten and Hyman [5] were the first to show how a Godunov scheme can be extended to handle moving grids in one-dimension, which improves the resolution of shocks and contact discontinuities. Subsequently, many studies on moving mesh methods for hyperbolic problems have been conducted. Berger and LeVeque proposed a mesh refinement method in which the mesh is refined locally based on some measure of the solution error using Cartesian sub-grids; this approach is especially successful in the case of higher-dimensional problems [6]. Stockie et al. developed an adaptive method that combines the flexibility and accuracy afforded by a dynamically moving mesh with the increased shock resolution capability of a Godunov-type scheme [7]. In particular, they constructed different monitor functions to capture shocks and discontinuities in different regions of physical solutions. One- and two-dimensional conservation laws were solved by Tang and Tang [8] using rezoning moving mesh methods that

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consist of two alternating parts: physical PDE time evolution and mesh redistribution. Further conservative interpolation was used to transfer the solution from the old mesh to the new mesh. This method has been shown to generally work well for hyperbolic conservation.

Only a few studies on time discretization have been conducted for moving mesh algorithms. Soheili and Salahshour studied dynamical moving mesh methods with local time-stepping techniques and applied the new method to blow-up problems [9]. Shu and Osher developed a total variation diminishing (TVD) Runge–Kutta method [10] for TVD high-order time discretization. The objective of high-order TVD Runge–Kutta time discretization is to maintain the TVD property while achieving higher-order accuracy in time. Such an approach has been shown to work very well for hyperbolic conservation laws [11]. The semi-discrete hyperbolic conservation laws and the discrete moving mesh equations form nonlinear differential algebraic equations [12], where the discrete moving mesh equations are the constraints. Thus far, to the best of our knowledge, there has been no discussion regarding the use of the TVD Runge–Kutta method in such a system, i.e., the coupling of the semi-discrete hyperbolic conservation laws with the discrete moving mesh equations. In fact, differential algebraic equations with constraints can be regularized as an index-two system [13,14]. Based on previous work, we propose a moving mesh algorithm with a Newton TVD Runge–Kutta scheme for the Euler equations, which improves time discretization in the moving mesh algorithms. The remainder of this paper is organized as follows. Section 2 describes the proposed method and presents the stability analysis. Section 3 provides some numerical examples. Finally, Section 4 summarizes our findings and concludes the paper with a brief discussion on the scope for future work.

2. Method

2.1. System

The Euler equations describing the evolution of an inviscid, compressible, polytropic gas in one-dimension are

$$\frac{\partial \mathbf{w}}{\partial t} + \frac{\partial}{\partial x}\mathbf{f}(\mathbf{w}) = 0, \quad t \geq 0, \quad x \in [a, b], \tag{1}$$

where

$$\mathbf{w} = \begin{pmatrix} \rho \\ \rho u \\ E \end{pmatrix}, \mathbf{f} = \begin{pmatrix} \rho u \\ \rho u^2 + p \\ (E+p)u \end{pmatrix}, \ E = \frac{1}{2}\rho u^2 + \frac{p}{\gamma - 1}. \tag{2}$$

In these equations, ρ is the density, u is the velocity, p is the pressure, and E is the total energy, and γ is the specific heat ratio. Further, x represents the physical coordinates and ξ is introduced as the physical coordinates. A one-to-one coordinate transformation from the computational domain Ω_c to the physical domain Ω_p is denoted by

$$x = x(\xi), \xi \in \Omega_c. \tag{3}$$

In practice, the uniform mesh ξ_j, $j = 0, 1, \ldots, N$, is given on the computational domain, where N is a certain positive integer, and the corresponding mesh points in the physical domain are given by

$$a \equiv x_0 < x_1 < x_2 < \cdots < x_{N-1} < x_N \equiv b. \tag{4}$$

Let $M(\xi, \bar{u})$ be a chosen positive monitor function that controls the moving mesh. Since the purpose is to achieve higher accuracy near the nonsmooth part of the solution, we introduce the monitor function of pseudo arc-length norm [15,16]

$$M(\xi, \bar{u}) = \sqrt{1 + \alpha\left(\frac{\partial \bar{u}}{\partial \xi}\right)^2}. \tag{5}$$

This monitor function places an emphasis on error control near rapid variation of the solution function. Here, α is the control coefficient and $\bar{u}$ is the chosen physical value. Following the integral form of the equidistribution principle [17,18], the transformation (3) satisfies

$$\int_a^{x(\xi)} M(s)ds = \xi \cdot \int_a^b M(s)ds. \tag{6}$$

Differentiating (6) with respect to ξ twice, we have

$$\frac{\partial}{\partial \xi}\left\{M\frac{\partial}{\partial \xi}x\right\} = 0 \tag{7}$$

Introducing an artificial time $\bar{t}$, in which the mesh relaxes toward equidistribution, we have the following equation of the moving mesh PDE:

$$\frac{\partial x}{\partial \bar{t}} = \frac{1}{\tau}\frac{\partial}{\partial \xi}\left(M\frac{\partial x}{\partial \xi}\right), \tag{8}$$