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Integral trees with diameter four

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ABSTRACT

A graph is called integral if all eigenvalues of its adjacency matrix consist entirely of integers. In this paper, we investigate integral trees $S(r; m_i) = S(a_1 + a_2 + \cdots + a_s; m_1, m_2, \ldots, m_s)$ of diameter 4 with $s = 3, 4, 5, 6$. Such integral trees are found by using a computer search or solving the Diophantine equations. New sufficient conditions for a construction of infinite families of integral trees $S(r'; m_i) = S(b_1 + \cdots + b_s; m_1, \ldots, m_s)$ of diameter 4 from given integral trees $S(r; m_i) = S(a_1 + \cdots + a_s; m_1, \ldots, m_s)$ of diameter 4 are given. Further, using these conditions we construct infinitely many new classes of integral trees $S(r'; m_i) = S(b_1 + \cdots + b_s; m_1, \ldots, m_s)$ of diameter 4 with $s = 3, 4, 5, 6$. Finally, we propose two basic open problems about integral trees of diameter 4 for further study.

1. Introduction

Let G be a simple graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G)$. The adjacency matrix $A(G)$ of G is defined as an $n \times n$ matrix $A(G) = (a_{ij})$, where $a_{ij} = 1$ if v_i is adjacent to v_j, and $a_{ij} = 0$ otherwise. The characteristic polynomial of G is the polynomial $P(G, x) = det(xI_n - A(G))$, where I_n denotes the $n \times n$ identity matrix. Let $\lambda_1 < \lambda_2 < \cdots < \lambda_t$ be t distinct eigenvalues of $P(G, x)$ with the corresponding multiplicities $k_1, k_2, \ldots, k_t$. The spectrum of $A(G)$ is also called the spectrum of G and denoted by $Spec(G) = \binom{\lambda_1\ \lambda_2\ \cdots\ \lambda_{t-1}\ \lambda_t}{k_1\ k_2\ \cdots\ k_{t-1}\ k_t}$.

We know that trees of diameter 4 can be formed by joining the centers of r stars $K_{1,m_1}, K_{1,m_2}, \ldots, K_{1,m_r}$ to a new vertex v. The tree is denoted by $S(r; m_1, m_2, \ldots, m_r)$ or simply $S(r;\ m_i)$. Assume that the number of distinct integers of $m_1, m_2, \ldots, m_r$ is s. Without loss of generality, assume that the first s ones are the distinct integers such that $0 \le m_1 < m_2 < \cdots < m_s$. Suppose that a_i is the multiplicity of m_i for each $i = 1, 2, \ldots, s$. The tree $S(r;\ m_i)$ is also denoted by $S(a_1 + a_2 + \cdots + a_s; m_1, m_2, \ldots, m_s)$, where $r = \sum_{i=1}^{s} a_i$ and $|V| = 1 + \sum_{i=1}^{s} a_i(m_i + 1)$. For all other facts on graph spectra (or terminology), see [7,8].

A graph G is called integral if all eigenvalues of its characteristic polynomial $P(G, x)$ are integers, that is, all eigenvalues of its adjacency matrix $A(G)$ consist entirely of integers. Since 1974, when Harary and Schwenk [11] proposed the question "Which graphs have integral spectra?", the research for integral graphs has been done (see the survey [3] and the Ph.D. thesis [23]). For some results about integral trees and integral graphs can be found in [3,4,6,10–14,18–20,23–29] and [1–3,7,8,11,22,23], respectively. It has been discovered recently [2,5,22] that integral graphs can play a role in the so-called

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perfect state transfer in quantum spin networks of quantum computing. In addition, there are also many results about some topological indices based on degree-based indices (e.g., Randić index [16,21]), distance-based indices (e.g., Wiener index [9], Wiener polarity index [17]) and spectrum-based indices (e.g., graph energy [15]).

In this paper, we investigate integral trees $S(r; m_i) = S(a_1 + a_2 + \cdots + a_s; m_1, m_2, \ldots, m_s)$ of diameter 4 with $s = 3, 4, 5, 6$. Such integral trees are found by using a computer search or solving the Diophantine equations. New sufficient conditions for a construction of infinite families of integral trees $S(r'; m_i) = S(b_1 + \cdots + b_s; m_1, \ldots, m_s)$ of diameter 4 from given integral trees $S(r; m_i) = S(a_1 + \cdots + a_s; m_1, \ldots, m_s)$ of diameter 4 are given. Further, using these conditions we construct infinitely many new classes of integral trees $S(r'; m_i) = S(b_1 + \cdots + b_s; m_1, \ldots, m_s)$ of diameter 4 with $s = 3, 4, 5, 6$. These results are a new contribution to the search of such integral trees. Finally, we propose two basic open problems about integral trees of diameter 4 for further study.

2. Preliminaries

In this section, we state some known results on integral trees of diameter 4.

Lemma 2.1 [14, 25]. *For the tree $S(r; m_i) = S(a_1 + \cdots + a_s; m_1, \ldots, m_s)$ of diameter 4, then we have*

$$P[S(r; m_i), x] = P[S(a_1 + \cdots + a_s; m_1, \ldots, m_s), x] = x^{1+\sum_{i=1}^{s} a_i(m_i-1)}$$
$$\cdot \prod_{i=1}^{s}(x^2 - m_i)^{a_i-1}\left[\prod_{i=1}^{s}(x^2 - m_i) - \sum_{i=1}^{s} a_i \prod_{j=1, j\neq i}^{s}(x^2 - m_j)\right].$$

Theorem 2.2 [26]. *The tree $S(r; m_i)$=$S(a_1 + a_2 + \cdots + a_s; m_1, m_2, \ldots, m_s)$ of diameter 4 is integral if and only if (i) $a_i = 1$ must hold if m_i is not a perfect square, (ii) all solutions of the following equation are integers.*

$$\sum_{i=1}^{s} \frac{a_i}{x^2 - m_i} = 1. \tag{1}$$

Moreover, there exist positive integers $u_1, u_2, \ldots, u_s$ satisfying

$$0 \le \sqrt{m_1} < u_1 < \sqrt{m_2} < u_2 < \cdots < u_{s-1} < \sqrt{m_s} < u_s < +\infty \tag{2}$$

such that the following linear equation system in $a_1, a_2, \ldots, a_s$ has positive integral solutions $(a_1, a_2, \ldots, a_s)$, and such that $a_i = 1$ must hold if m_i is not a perfect square.

$$\begin{cases} \dfrac{a_1}{u_1^2 - m_1} + \dfrac{a_2}{u_1^2 - m_2} + \cdots + \dfrac{a_s}{u_1^2 - m_s} = 1, \\ \cdots\cdots\cdots \quad \cdots\cdots\cdots \\ \dfrac{a_1}{u_s^2 - m_1} + \dfrac{a_2}{u_s^2 - m_2} + \cdots + \dfrac{a_s}{u_s^2 - m_s} = 1. \end{cases} \tag{3}$$

Theorem 2.3 [13, 29]. *The tree $S(r; m_i) = S(a_1 + \cdots + a_s; m_1, \ldots, m_s)$ of diameter 4 is integral if and only if there exist positive integers u_i and nonnegative integers m_i $(i = 1, 2, \ldots, s)$ such that $0 \le \sqrt{m_1} < u_1 < \sqrt{m_2} < u_2 < \cdots < u_{s-1} < \sqrt{m_s} < u_s < +\infty$, and such that*

$$a_k = \frac{\prod_{i=1}^{s}(u_i^2 - m_k)}{\prod_{i=1, i\neq k}^{s}(m_i - m_k)}, \quad (k = 1, 2, \ldots, s) \tag{4}$$

are positive integers, and such that $a_i = 1$ must hold if m_i is not a perfect square. (Note that u_i are eigenvalues of the tree).

Corollary 2.4 [28, 29]. *If the tree $S(r; m_i) = S(a_1 + a_2 + \cdots + a_s; m_1, m_2, \ldots, m_s)$ of diameter 4 is integral, then we have the following results.*

1. $a_1 > 1$. *Moreover m_1 is a perfect square.*
2. $r = \sum_{i=1}^{s} a_i = \sum_{i=1}^{s} u_i^2 - \sum_{i=1}^{s} m_i$.

Theorem 2.5 [26]. *If $m_1(\ge 0), m_2, \ldots, m_s$ are perfect squares, then the tree $S(a_1 + a_2 + \cdots + a_s; m_1, m_2, \ldots, m_s)$ of diameter 4 is integral if and only if the tree $S(a_1n^2 + a_2n^2 + \cdots + a_sn^2; m_1n^2, m_2n^2, \ldots, m_sn^2)$ is integral for any positive integer n.*

Remark 2.6 (see also [26]). Let $GCD(p_1, p_2, \ldots, p_s)$ denote the greatest common divisor of the numbers $p_1, p_2, \ldots, p_s$. For $m_1(\ge 0), m_2, \ldots, m_s$ are perfect squares, we say that a vector $(a_1, a_2, \ldots, a_s, m_1, m_2, \ldots, m_s)$ is *primitive* if $GCD(a_1, a_2, \ldots, a_s, m_1, m_2, \ldots, m_s) = 1$. Theorem 2.5 shows that it is reasonable to study Eq. (3) only for primitive vector $(a_1, a_2, \ldots, a_s, m_1, m_2, \ldots, m_s)$.