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Computational implementation of the inverse continuous wavelet transform without a requirement of the admissibility condition

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ABSTRACT

Recently, it has been proven Lebedeva and Postnikov (2014) that the continuous wavelet transform with non-admissible kernels (approximate wavelets) allows an existence of the exact inverse transform. Here, we consider the computational possibility for the realization of this approach. We provide a modified simpler explanation of the reconstruction formula, restricted on the practical case of real valued finite (or periodic/periodized) samples and the standard (restricted) Morlet wavelet as a practically important example of an approximate wavelet. The provided examples of applications include the test function and the non-stationary electro-physical signals arising in the problem of neuroscience.

1. Introduction

The continuous wavelet transform (CWT) with the standard (restricted) Morlet wavelet $\psi(\xi) = \exp\left(i\omega_0\xi - \xi^2/2\right)$ is defined as the integral

$$w(a, b) = \int_{-\infty}^{+\infty} f(t) e^{-i\omega_0 \frac{t-b}{a}} e^{-\frac{(t-b)^2}{2a^2}} \frac{dt}{\sqrt{2\pi a^2}}, \tag{1}$$

where a and b are the scale and the shift correspondingly, ω_0 is the central frequency.

It is one of the most powerful modern tools of signal processing especially adjusted to the extraction of instant oscillating patterns [1,2] since the transform (1) of the harmonic oscillation $f(t) = \exp(i\omega t)$ results in the complex function

$$w(a, b) = e^{i\omega b} e^{-\frac{(a\omega-\omega_0)^2}{2}}.$$

The modulus of this function has a maximum, which allows for the determining of signal's frequency $\omega = \omega_0/a_{max}$ and the phase coincides with the signal's one.

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The modern applications of the continuous wavelet transform are focused, in particular, on a study of environmental time series [3,4], geo- and astrophysics [5–7], biophysics [8–10] and neuroscience, see an extensive review in the recently published book [11].

At the same time, the actual problem is not restricted by the search of oscillating patterns localizations. It is important to extract revealed structures from the background consisting of a noise, global oscillations and inhomogeneities, etc. [12–15].

However, the conventional approaches to the inversion of the wavelet transform with the standard Morlet wavelet have some principal difficulties from the point of view of functional analysis. To be applicable in a classical inversion formula, a wavelet function ψ should satisfy the admissibility condition [1]

$$C_\psi = \int_{\mathbb{R}} \frac{|\psi(\omega)|^2}{|\omega|} d\omega < \infty,$$

where the asterisk denotes a complex conjugation. Then the classical inversion of the wavelet transform $W_\psi f = w(a, b)$ with a wavelet function ψ is written as

$$f(t) = \frac{1}{C_\psi} \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{1}{a} \psi\left(\frac{t-b}{a}\right) w(a, b) \frac{dadb}{a}.$$

However, the integral C_ψ diverges for the standard (restricted) Morlet wavelet.

On the other hand, the alternative inversion formula for the CWT

$$\frac{1}{\pi} \text{v.p.} \int_{\mathbb{R}} \frac{db}{b-t} \int_{\mathbb{R}} \frac{\partial}{\partial b} w(a, b)\, da = \psi^*(0)\, f(t) \tag{2}$$

is proven in [16] under a mild natural conditions on a wavelet function ψ.

The principal aim of this paper is to show how this approach can be realized in applications. For this reason, we adapt the proof of (2) to the case of real-valued functions with a finite support in such a way that its line of reasoning and the result can be straightforwardly used as a practical algorithm for the reconstruction of a function from its wavelet transform with the Morlet wavelet.

The paper is organized as follows. Section 2 shortly discusses the implementation of the CWT with the Morlet wavelet in the application to real-valued functions determined on a finite interval (or periodic on $\mathbb{R}$) and then presents the simple proof the exact reconstruction formula that does not require the wavelet admissibility condition. Its formulation allows the simple computational realization, which is presented in Section 3. The examples comprise a test function and the solution of a modern practical problem in the field of neuroscience. Neuronal electric activity is characterized by the high non-linearity and complexity. Therefore, the localization of extracted features is very important for revealing of interactions between different cells in neuronal networks [17]. The final section gives some outlooks for the perspectives for the applications of the proposed method in computational physics.

2. Direct and inverse continuous wavelet transform of periodic functions

2.1. CWT expansion

Let us consider a real-valued function $f(t)$ with the zero mean given by its expansion into the Fourier series

$$f(t) = \sum_{n=1}^{\infty} A_n \cos(\omega_n t) + B_n \sin(\omega_n t), \tag{3}$$

where $\omega_n = 2\pi n/T$.

This function is periodic with the period T. As well, it can be considered as a function determined within the interval $t \in [0, T]$ and periodically extended over all $\mathbb{R}$. The last point of view is applicative in practical computations since one can deal with finite samples only. The assumption of the zero mean is also not restrictive for the problems of computational physics since it can be easily achieved by the exclusion of the averaged value: $f(t) - T^{-1} \int_0^T f(t) dt \to f(t)$.

For the wavelet analysis of spectral components, it is most convenient to operate with the complex analytical counterpart of the function $f(t)$ obtained via the Hilbert transform $f_a(t) = f(t) + iH[f(t)]$.

Since $H[\cos(x)] = \sin(x)$, $H[\sin(x)] = -\cos(x)$, the Hilbert transform applied to Eq. (3), accompanied with Euler's formula, gives

$$f_a(t) = \sum_{n=1}^{\infty} (A_n - iB_n) e^{i\omega_n t} = \sum_{n=1}^{\infty} C_n e^{i\omega_n t}. \tag{4}$$

Note also that Euler's formula applied directly to Eq. (3) results in the representation

$$f(t) = \frac{1}{2} \sum_{n=1}^{\infty} C_n e^{i\omega_n t} + C_n^* e^{-i\omega_n t},$$

where asterisk denotes the complex conjugation.