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Dynamics of a stochastic SIR epidemic model with saturated incidence

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ABSTRACT

In this paper, the dynamics of a stochastic SIR epidemic model with saturated incidence is investigated. Firstly, we prove that the system has a unique global positive solution with any positive initial value. Then we verify that random effect may lead the disease to extinction under a simple condition. Thirdly, we establish a sufficient condition for persistence in the mean of the disease. Moreover, we show that there is a stationary distribution to the stochastic system under certain parametric restrictions. Finally, some numerical simulations are carried out to confirm the analytical results.

1. Introduction

The SIR epidemic model is one of the most important models in epidemiological patterns and disease control. Kermack and McKendrick [1] initially proposed and investigated the classical SIR (susceptible-infected-removed) model. From then on, many authors have studied the SIR epidemic model (see e.g. [2–5]).

The deterministic SIR model can be expressed by the following ordinary differential equations:

$$\begin{cases} S'(t) = \Lambda - \beta S(t)I(t) - \mu S(t), \\ I'(t) = \beta S(t)I(t) - (\mu + \gamma + \epsilon)I(t), \\ R'(t) = \gamma I(t) - \mu R(t), \end{cases} \tag{1.1}$$

where $S(t)$ denotes the number of the individuals susceptible to the disease, $I(t)$ is the number of the individuals who are infectious and $R(t)$ represents the members who have been removed from the possibility of infection. Λ denotes the influx of individuals into the susceptible, μ represents the natural death rate of S, I, R compartments, γ denotes the recovery rate of the infective individuals, β represents the transmission rate and ϵ denotes the death rate due to disease. Λ, β, μ, γ, ϵ are all positive. In system (1.1), the basic reproduction number is $R_0 = \frac{\beta\Lambda}{\mu(\mu+\gamma+\epsilon)}$. System (1.1) always has a disease-free equilibrium $E_0(\frac{\Lambda}{\mu}, 0, 0)$. When $R_0 \le 1$, the disease-free equilibrium E_0 is globally asymptotically stable, which means that the disease will die out. When $R_0 > 1$, then E_0 is unstable and there exists a globally asymptotically stable endemic equilibrium

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$E_* = (S_*, I_*, R_*)$, where $S_* = \frac{\Lambda}{\mu R_0}$, $I_* = \frac{\mu}{\beta}(R_0 - 1)$, $R_* = \frac{\gamma}{\beta}(R_0 - 1)$, which means that the disease will prevail and persistent in the population. These conclusions of system (1.1) can be found from [6].

It is well known that the incidence rate of a disease plays an important role in the investigation of mathematical epidemiology. In many previous epidemic models, the bilinear incidence rate βSI is frequently used (see e.g. [7–11]). However, as the number of susceptible individuals becomes large, due to the number of susceptible individuals with which every infective contacts within a certain time being limited, it is unreasonable to adopt the bilinear incidence rate. If the number of partners becomes large, there exists a saturation influence. These facts show that it is reasonable to adopt the saturated incidence and the saturated incidence may be more realistic than bilinear incidence (see e.g. [12–14]).

To make system (1.1) more realistic and interesting, in this paper, we adopt the saturated incidence rate $\frac{\beta SI}{1+\alpha I}$, then the SIR epidemic model with saturated incidence takes the following form:

$$\begin{cases} S'(t) = \Lambda - \dfrac{\beta S(t)I(t)}{1+\alpha I(t)} - \mu S(t), \\ I'(t) = \dfrac{\beta S(t)I(t)}{1+\alpha I(t)} - (\mu + \gamma + \epsilon)I(t), \\ R'(t) = \gamma I(t) - \mu R(t), \end{cases} \tag{1.2}$$

where α is a positive constant and other parameters are defined as system (1.1).

As a matter of fact, epidemic systems are always affected by environmental noise which can provide an additional degree of realism in compared to their corresponding deterministic models. The reason is that environmental fluctuation like weather fluctuation, infectious disease etc are responsible to consider stochastic perturbation in the deterministic models. Thus, it is an important component in an ecosystem. May [15] has revealed that due to environmental fluctuation, the birth rates, death rates, transmission coefficient and other parameters involved with the system exhibit random fluctuations to a greater or lesser extent. Consequently, many researchers introduced stochastic perturbations into deterministic models to reveal the effects of environmental noises on the epidemic models (see e.g. [16–21]).

As an extension of system (1.2), we introduce random perturbation into model (1.2) by replacing the parameter β by $\beta + \sigma\dot{B}(t)$, where $\dot{B}(t)$ is the white noise, namely, $B(t)$ is a standard Brownian motion and σ^2 denotes the intensity of $B(t)$. Here we adopt the standard Brownian motion to model the stochastic perturbation because the standard Brownian motion incorporates some sort of vital data. The reason is that since the model parameters completely describe the environmental phenomena, with the change of environment, the model parameters will also change. Hence, to take into account the environmental fluctuation in the system, we consider the model parameter as random, which can be done by perturbing the model parameter by white noise terms that is nothing but the time derivative of standard Brownian motion. Then we obtain the following stochastic SIR epidemic model:

$$\begin{cases} dS(t) = \left(\Lambda - \dfrac{\beta S(t)I(t)}{1+\alpha I(t)} - \mu S(t)\right)dt - \dfrac{\sigma S(t)I(t)}{1+\alpha I(t)}dB(t), \\ dI(t) = \left(\dfrac{\beta S(t)I(t)}{1+\alpha I(t)} - (\mu + \gamma + \epsilon)I(t)\right)dt + \dfrac{\sigma S(t)I(t)}{1+\alpha I(t)}dB(t), \\ dR(t) = (\gamma I(t) - \mu R(t))dt. \end{cases} \tag{1.3}$$

Since the dynamic of R has no influences on the transmission dynamics, we can omit to analyze the third equation of system (1.3) and only investigate the following system:

$$\begin{cases} dS(t) = \left(\Lambda - \dfrac{\beta S(t)I(t)}{1+\alpha I(t)} - \mu S(t)\right)dt - \dfrac{\sigma S(t)I(t)}{1+\alpha I(t)}dB(t), \\ dI(t) = \left(\dfrac{\beta S(t)I(t)}{1+\alpha I(t)} - (\mu + \gamma + \epsilon)I(t)\right)dt + \dfrac{\sigma S(t)I(t)}{1+\alpha I(t)}dB(t). \end{cases} \tag{1.4}$$

The main aim of this paper is to investigate the dynamics of system (1.4).

The organization of this paper is as follows: In Section 2, the global existence and positivity of the solution to system (1.4) are investigated. In Section 3, we show that random effect may lead the disease to extinction under a simple condition. In Section 4, we establish a sufficient condition for persistence in the mean of the disease. In Section 5, by choosing suitable Lyapunov functions, we prove that system (1.4) has a unique stationary distribution under certain parametric restrictions. Section 6 is devoted to introducing some numerical simulations to illustrate our theoretical results. Finally, some conclusions and discussions are given.

Throughout this paper, unless otherwise specified, let $(\Omega, \mathcal{F}, \mathcal{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions. We also let $\mathbb{R}^d_+ = \{x \in \mathbb{R}^d : x_i > 0, 1 \leq i \leq d\}$ and $B(t)$ represent a standard Brownian motion defined on this probability space.