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On ordering of complements of graphs with respect to matching numbers

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ABSTRACT

We consider an earlier much studied quasi-order, defined in terms of matching numbers of graphs, and apply it to graph complements. We establish four transformations on the complements of graphs that increase or decrease the matching numbers accordingly. Several applications of these results are put forward.

1. Introduction

All graphs considered in this paper are undirected and simple. Let $G = (V(G), E(G))$ be such a graph, with vertex set $V(G)$ and edge set $E(G)$. A matching of G is a set of pairwise non-adjacent edges in $E(G)$. A k-matching is a matching consisting of k edges. By $m(G, k)$ we denote the number of k-matchings of G [7,40]. It is both consistent and convenient to define $m(G, 0) = 1$ as well as $m(G, k) = 0$ for $k < 0$ and $k > n/2$, where $n = |V(G)|$ is the order of G.

Many results pertaining to matching numbers could be expressed by means of the *matching polynomial* [9], which is usually defined as

$$\alpha(G, \lambda) = \sum_{k\geq 0}(-1)^k\, m(G, k)\, \lambda^{n-2k}\,.$$

Details of the theory of matching polynomial can be found in the monographs [7,16].

There is a natural ordering with respect to the matching numbers. If for two graphs G_1 and G_2 the relations $m(G_1, k) \geq m(G_2, k)$ are satisfied for all k, then we write $G_1 \succeq G_2$ (or $G_2 \preceq G_1$). If $G_1 \succeq G_2$ and $m(G_1, k) > m(G_2, k)$ for some k, then we write $G_1 \succ G_2$ (or $G_2 \prec G_1$). If both $G_1 \succeq G_2$ and $G_2 \succeq G_1$ hold, then we write $G_1 \sim G_2$ and say that G_1 and G_2 are matching equivalent.

As a binary relation on graphs, $\preceq$ is reflexive and transitive, but not anti-symmetric because there are non-isomorphic graphs G_1 and G_2 such that $G_1 \sim G_2$. Hence $\preceq$ is a quasi-order. Since there exist graphs for which neither $G_1 \succeq G_2$ nor $G_2 \succeq G_1$ holds, which means that G_1 and G_2 are *incomparable* w.r.t. the relations $\succeq$, the ordering implied by this relation is not complete.

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The above specified quasi-order was introduced in the 1970s by one of the present authors [12,13]. Since then, it was extensively used, especially in connection with the energy of trees [19,31], for which the relation

$$E(T) = \frac{2}{\pi}\int_0^{\infty}\frac{1}{x^2}\ln\left[\sum_{k\geq 0} m(T,k)\,x^{2k}\right]dx\,. \tag{1}$$

was shown to hold [12]. It is not difficult to observe that the integral on the right hand side of Eq. (1) is increasing in all the coefficients $m(G, k)$. From Eq. (1), it immediately follows that if $T_1 \succeq T_2$ holds for two trees T_1 and T_2, then $E(T_1) \geq E(T_2)$. For recent applications of this property of trees see [6,34,35,37,39]. If the pairs of graphs are incomparable, then the considerations based on Eq. (1) are much more complicated; for details see [24,25].

Another straightforward application of the quasi-order is for comparing Hosoya indices. The Hosoya index of a graph G is defined as the total number of matchings in G, i.e., as $Z(G) = \sum_{k\geq 0} m(G,k)$; for details and further references see [32,41,47]. At this point it is worth noting that via the Hosoya index, the matching numbers $m(G, k)$ have been related also to certain types of entropy [8,22,30,36].

In 2012, Wagner and one of the present authors [20] extended the applicability of formula (1) to all graphs, by conceiving the concept of *matching energy*, defined as

$$ME(G) = \frac{2}{\pi}\int_0^{\infty}\frac{1}{x^2}\ln\left[\sum_{k\geq 0} m(G,k)\,x^{2k}\right]dx\,. \tag{2}$$

Evidently, if G is a tree, then $ME(G) = E(G)$. The matching energy is nowadays the subject of extensive studies, see the survey [17], the recent papers [1–5,23,27,29,42,44], and the references cited therein.

Thus, we observe that

$$\begin{aligned} &\text{If } G \succeq H\,,\text{ then } \;Z(G) \geq Z(H) \;\text{ and }\; ME(G) \geq ME(H)\\ &\text{If } G \succ H\,,\text{ then } \;Z(G) > Z(H) \;\text{ and }\; ME(G) > ME(H)\\ &\text{If } G \sim H\,,\text{ then } \;Z(G) = Z(H) \;\text{ and }\; ME(G) = ME(H)\,. \end{aligned}$$

The quasi-order $\succeq$ was studied for various classes of graphs: acyclic [12,13], unicyclic [14], bicyclic [14,15], tricyclic [18], and many others [21,38,43,45]. For these classes the maximal and minimal elements with respect to $\succeq$ could be determined. In particular, the maximum and minimum elements in the class of connected graphs with n vertices are the complete graph K_n and the star S_n, respectively [20]. So and Wang [38] determined the minimum elements among all connected graphs of order n and size m for $n-1 \leq m \leq 2n-3$ and $\frac{n(n-1)}{2} - (n-2) \leq m \leq \frac{n(n-1)}{2}$.

In this paper, we first present some transformations that increase or decrease the matching numbers of the complements of graphs. Then several applications are put forward.

2. Main results

We start by providing a few necessary definitions and auxiliary lemmas.

A path $P = (v_0, e_1, v_1, \ldots, v_{p-1}, e_p, v_p)$ in a graph H is said to be a *pendent path* (starting from v_0) of length p, if all the vertices $v_1, \ldots, v_{p-1}$ are of degree two, and the vertex v_p is of degree one. Let G be a graph and u be one of its vertices that is not an isolated vertex. Let $G(u;\, a,\, b)$ denote the graph obtained from G by attaching two pendent paths of length a and b at the vertex u. Then we say that $G(u; a+b, 0)$ is obtained from $G(u;\, a,\, b)$ by a *total grafting* operation at u.

For a graph G and v a vertex of G, by $N_G(v)$ is denoted the set of vertices in G adjacent to v.

Lemma 1. [7, 16] *If $e = uv$ is an arbitrary edge of G with end vertices u and v, then for all non-negative integers k,*

$$\begin{aligned} m(G,k) &= m(G-e,k) + m(G-u-v,k-1)\\ m(G,k) &= m(G-u,k) + \sum_{w\in N_G(u)} m(G-u-w,k-1)\,. \end{aligned}$$

Let G and H be two graphs whose vertex sets are disjoint. If v is a vertex of G and w a vertex of H, let $G(v, w)H$ denote the graph obtained by identifying the vertices v and w.

Lemma 2. [11] *Let G and H be graphs with disjoint vertex sets. For v and w being vertices of G and H, respectively, and for all k,*

$$m(G(v,w)H,k) = m(G\cup H-w,k) + m(G-v\cup H,k) - m(G-v\cup H-w,k)\,.$$

As usual, by $\overline{G}$ we denote the complement of the graph G. In the theory of matching polynomials it has been shown that the matching polynomial of $\overline{G}$ can be computed from the matching polynomial of the graph G. Let K_p be the complete graph of order p. Then, the following holds: