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Tail variance of portfolio under generalized Laplace distribution

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ABSTRACT

Many popular downside risk measures such as tail conditional expectation and conditional value-at-risk only characterize the tail expectation, but pay no attention to the tail variance beyond the value-at-risk. This is a severe deficiency of risk management in finance and insurance industry, especially in measuring extreme risk with large losses. We derive the explicit formulae of the tail variance of portfolio under the assumption of generalized Laplace distribution, and mixture generalized Laplace distribution as well. Some numerical results of parameters related to the tail variance of portfolio are also provided. Finally, we present an example of application to optimization tail mean-variance portfolio. The empirical results show that the performance of optimal portfolio can be efficiently improved by controlling the tail variability of returns distribution.

1. Introduction

It is well recognized that the measure of risk has a crucial role in portfolio optimization under uncertainty. Value-at-risk (VaR), developed by JP Morgan, is one of the most popular risk measure widely employed by financial institutions. During the last decades, VaR has been become a standard risk measure written into industry regulations due to its conceptual simplicity, ease of computation and ready applicability. However, VaR also received severe criticisms from practitioners and researchers because it has undesirable mathematical characters such as lack of subadditivity, a notation in coherent risk measure framework initiated by Artzner et al. [3], see also [1,37] for the very definition. VaR has also been charged for being too conservative, in that it takes no account of the magnitude of extreme loss beyond the VaR quantile. In fact, random loss variables with light-tailed distributions or heavy-tailed distributions may have the same VaR quantile (see [12]). But they clearly do not have the same behavior in their extremes. Other criticisms of VaR come from [2], who argued that the VaR is in general non-convex. As a result, the problem of minimizing VaR of a portfolio can have multiple local minimizes.

To remedy the shortcoming inherent in VaR, Artzner et al. [3] proposed the use of expected shortfall, or ES for short, which is defined as the conditional expectation of loss beyond the VaR level. Other alternative measures include conditional value-at-risk (CVaR) in [33], tail conditional expectation (TCE) in [3], worst conditional expectation (WCE) in [4], conditional drawdown-at-risk (CDaR) in [41], tail standard deviation (TSD) in [13], and translation-invariant and positive homogeneous (TIPH) risk measure in [23,24]. It is noteworthy that CVaR as a coherent risk measure coincides with ES and TCE under the assumption of continuity distribution, and CVaR also is rather appealing in portfolio optimization due to its attractive

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properties such as convexity and continuity with respect to portfolio weights, see [15,40,43] for example. Nevertheless, CVaR is also criticized by Lim et al. [28] for being fragile in portfolio optimization due to estimation errors.

While the risk measures mentioned above undoubtedly make remarkable improvement on VaR, they only are the linear probability weighted combination of losses beyond VaR. Some researchers argue that the higher orders of moments of the loss distribution should be considered in order to control and decrease the extreme loss. For example, Cheng and Wang [10] proposed a new coherent risk measure based on p-norms with application in realistic portfolio optimization, see [8,9] for further development. Again for instance, Valdez [38] examined the tail conditional variance (hereafter, TCV) for some common parametric families of distributions. Landsman [22] proposed the tail variance (TV) risk defined by the variance of the loss distribution beyond some critical value. They also established correspondingly a tail mean-variance portfolio selection model. Owadally and Landsman [30] further discussed the character of optimal portfolio under tail mean-variance criterion. Recently, El Methni et al. [12] introduced a new risk measure, the so-called conditional tail moment (CTM), which unifies all risk measures based on conditional moments of the loss variable beyond VaR such as TCE, TCV and CVaR, and they proposed a nonparametric approach to estimating these risk measures, see also [11].

It is worth noting that, much more attention has been paid to the TV risk measure because it considers the variability of the risk along the tail of distribution, see [25,35] and references therein. There has also been growing interesting in studying the evaluation the TV risk and its application to portfolio and capital allocation. For example, Furman and Landsman [13] investigated the TV risk measure under elliptical contoured distribution. Landsman et al. [25] derived formulae for the tail variance and the tail variance premium (TVP) of risks in a multivariate log-elliptical setting. Ignatieva and Landsman [17] investigated TCE and TVP for the family of symmetric generalized hyperbolic distribution. Xu and Mao [39] proposed a novel capital allocation rule based on the tail mean-variance principle.

Motivated by the above discussion and Kamdem [18–20], in which the author derived the closed-form expression of VaR and ES of linear portfolio under generalized elliptical distribution, generalized Laplace distribution and their mixture distributions respectively, in this paper, we will study the explicit expression of TV risk of linear portfolios with assets returns governed by a generalized Laplace distribution or the so-called generalized error distribution (GED), and a mixture of generalized Laplace distributions as well. Kmadem [20] suggested that generalized Laplace distribution is a considerable improvement over student t-distribution. The main shortcoming of the multivariate t-distribution is that all the marginal distributions must have the same degree of freedom, which implies that every risk factor has equally heavy tails. See also [29] for an empirical comparison between the t-distribution and the GED distribution. Additional important examples of heavy distribution are, for example, mixture normal distribution in [7], asymmetric t-distribution in [42], skew-normal distribution in [5], skew t-distribution in [14]. The overall review of such distribution is actually beyond the scope of the this paper, and more discussions are available in [6,21,36] and references therein.

The remainder of the paper is organized as follows. Section 2 presents the explicit formulae of the TV risk of portfolio under the generalized Laplace distribution. Section 3 extends the explicit formulae of the TV risk of portfolio to the case of mixture generalized Laplace distribution. Section 4 presents an application to tail mean-variance optimal portfolio and an empirical example from Chinese stock market. After brief concluding remarks in section 5, an appendix collects all mathematical proofs of the paper.

2. Tail variance of portfolio under generalized Laplace distributions

2.1. Some notations and downside risk measure

In this section, we introduce some notations and familiar quantile risk measures. Let $n \geq 2$ be the number of risky asset in the economy. The random rate of return of the jth asset is denoted by r_i , and the rate of returns vector is denoted by $r = (r_1, r_2, \ldots, r_n)'$. Let μ and Σ be the expectation and variance–covariance matrix of r respectively. The investment weight vector $\omega = (\omega_1, \omega_2, \ldots, \omega_n)'$ is called a portfolio, where ω_i is the fraction of wealth invested in asset i. The rate of return of portfolio is formulated by $r_\omega = r'\omega$. Let $X = -r_\omega$ stand for the loss of the portfolio. Denote $\mu_\omega = \mu'\omega$ and $\sigma^2_\omega = \omega'\Sigma\omega$ as the expected rate of returns and variance of the rate of returns of the portfolio ω, respectively.

Now we recall some downside risks based on quantile such as value-at-risk (VaR), tail conditional expectation (TCE) and expected shortfall (ES). The VaR and TCE of the portfolio at confidence level $1-\alpha$ are defined as

$$VaR_\alpha(X) = \inf\{x | P(X \leq x) \geq 1-\alpha\},$$
$$TCE_\alpha(X) = E\big[X | X > VaR_\alpha(X)\big].$$

This notation is closely related to ES and CVaR, because all of them, from the viewpoint of definition, describe the average loss that is greater than the VaR level, despite they are in general not coincident unless suitable condition, such as continuous probability distribution of returns, is satisfied. However, we note that these risk measures only describe the mean of tail loss beyond VaR, but pay no attention to the deviation of such tail loss. This is obviously inadequate for measuring tail risk, especially for the loss of extreme event with small probability but large loss amounts (see e.g., [16,34]). To describe the variability of the tail loss and consequently better control the extreme risk, Valdez [38] proposed a tail conditional variance (TCV) defined as

$$TCV_\alpha(X) = E\big[(X - E(X))^2 | X > VaR_\alpha(X)\big].$$