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A biparametric extension of King's fourth-order methods and their dynamics

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ABSTRACT

A class of two-point quartic-order simple-zero finders and their dynamics are investigated in this paper by extending King's fourth-order family of methods. With the introduction of an error corrector having a weight function dependent on a function-to-function ratio, higher-order convergence is obtained. Through a variety of test equations, numerical experiments strongly support the theory developed in this paper. In addition, relevant dynamics of the proposed methods is successfully explored for a prototype quadratic polynomial as well as parameter spaces and dynamical planes.

1. Introduction

In many scientific and technological fields, we have naturally encountered root-finding problems for nonlinear equations of the form $f(x) = 0$. Exact solutions of the given governing equations are available in limited special cases. In most cases, however, only approximate solutions may resolve the actual problems under consideration. Among many simple-zero finders, we widely employ the classical Newton's method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, n = 0, 1, 2, \dots \tag{1.1}$$

that solves $f(x) = 0$ without difficulty, provided that a good initial guess x_0 is chosen near the zero α. It is known that numerical scheme (1.1) is a second-order one-point optimal [29] method on the basis of Kung–Traub's conjecture [29] that any multipoint method [26,27,36] without memory can reach its convergence order of at most 2^{r-1} for r functional evaluations. Other higher-order methods for nonlinear equations are referred in Section 2.

This paper is divided into seven sections. Following this introductory section, Section 2 shortly describes existing studies on simple-zero finders. Investigated in Section 3 is methodology and convergence analysis for newly proposed simple-zero finders. A main theorem is established to state convergence order of four as well as to derive asymptotic error constants and error equations by use of a family of weight functions W_f dependent on a function-to-function ratio. In Section 4, special forms of the second-stage error correctors (to be defined later) are considered with weight functions of polynomial and rational types of functions. Section 5 discusses the dynamics behind the fixed points of the proposed iterative maps. Extensively investigated are dynamical properties of the proposed methods along with illustrative description on stability

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analysis of their fixed points, parameter spaces and convergence planes. Tabulated in Section 6 are computational results for a variety of numerical examples. Table 5 compares the magnitudes of $e_n = x_n - \alpha$ of the proposed methods with those of typical existing methods. Section 7 states the overall conclusion and briefly discusses possible future work enhancing the current approach.

2. Review of existing simple-zero finders

Fourth-order simple-root finders for a given nonlinear equation $f(x) = 0$ have been developed by researchers such as Argyros et al. [8], Chun [17,19], Ghanbari [23], Kou et al. [28], Maheshwari [33] and Sharma et al. [37] including many other researchers [15,22,26,27]. Special attention is paid to two-point Jarratt's method [8] with optimal order of four shown below by (2.1):

$$\begin{cases} y_n = x_n - \frac{2}{3}\cdot\frac{f(x_n)}{f'(x_n)}, \\ x_{n+1} = x_n - Q(x_n)\cdot\frac{f(x_n)}{f'(x_n)}, \end{cases} \tag{2.1}$$

where $Q(x_n) = 1 - \frac{3}{2}\cdot\frac{f'(y_n)-f'(x_n)}{3f'(y_n)-f'(x_n)} = \frac{1}{2}\cdot\frac{3f'(y_n)+f'(x_n)}{3f'(y_n)-f'(x_n)}$. One should note that Jarratt's method is a two-point method with evaluations of two derivatives and one function.

Via uniparametric generalization of (2.1), Chun [17] developed another family of optimal fourth-order methods as shown below:

$$\begin{cases} y_n = x_n - \frac{2}{3}\cdot\frac{f(x_n)f'(x_n)}{f'(x_n)^2-\lambda f(x_n)}, \\ x_{n+1} = x_n - Q(x_n)\cdot\frac{f(x_n)}{f'(x_n)}, \end{cases} \tag{2.2}$$

where $Q(x_n) = 1 + \frac{1}{2}\cdot\frac{(f'(x_n)-f'(y_n))(f'(x_n)^2-\lambda f(x_n))}{\frac{2}{3}f'(x_n)^3-(f'(x_n)-f'(y_n))(f'(x_n)^2-\lambda f(x_n))}$ with $\lambda \in \mathbb{R}$. One should observe that (2.2) reduces to (2.1) when $\lambda = 0$.

By another extension of (2.1), Kou et al. [28] developed a two-parameter family of optimal fourth-order methods as shown below:

$$\begin{cases} y_n = x_n - \frac{2}{3}\cdot\frac{f(x_n)}{f'(x_n)}, \\ x_{n+1} = x_n - Q(x_n)\cdot\frac{f(x_n)}{f'(x_n)}, \end{cases} \tag{2.3}$$

where $Q(x_n) = 1 - \frac{3}{4}\left(1 - \frac{(\frac{3}{2}-a)(f'(y_n)-f'(x_n))}{bf'(y_n)+(1-b)f'(x_n)}\right)\cdot\frac{f'(y_n)-f'(x_n)}{af'(y_n)+(1-a)(f'(x_n)}$ with $a, b \in \mathbb{R}$. Note that (2.3) reduces to (2.1) when $a = b = 3/2$

Convergence behavior of existing methods (2.1)–(2.3) for various test equations will be compared later in Section 6 with proposed methods to be investigated in the next section.

3. Methodology and convergence analysis

Let a function $f : \mathbb{C} \to \mathbb{C}$ have a simple zero α and be analytic [1,25] in a small neighborhood of α. Then, given an initial guess x_0 sufficiently close to α, we propose in this paper a family of new two-point optimal fourth-order simple-zero finders of the form:

$$\begin{cases} y_n = x_n - \frac{h}{(1+\lambda h)}, h = \frac{f(x_n)}{f'(x_n)}, \\ x_{n+1} = y_n - W_f(u)\cdot\frac{h}{(1+\rho h)}, W_f(u) = \frac{u(a+bu)}{1+cu}, u = \frac{f(y_n)}{f(x_n)}, \end{cases} \tag{3.1}$$

where $\lambda, \rho, a, b, c \in \mathbb{C}$ are parameters to be determined for optimal order of convergence. We find that $u = O(h) = O(\frac{h}{(1+\lambda h)}) = O(\frac{h}{(1+\rho h)}) = O(e_n)$.

Definition 1.1 (Error equation, asymptotic error constant, order of convergence)**.** Let $x_0, x_1, \ldots, x_n, \ldots$ be a sequence converging to α and $e_n = x_n - \alpha$ be the nth iterate error. If there exist real numbers $p \in \mathbb{R}$ and $b \in \mathbb{R} - \{0\}$ such that the following error equation holds

$$e_{n+1} = b{e_n}^p + O\left(e_n^{p+1}\right), \tag{3.2}$$

then b or $|b|$ is called the asymptotic error constant and p is called the order of convergence [38].

In this paper, we investigate the maximal convergence order of proposed methods (3.1). We here establish a main theorem describing the convergence analysis regarding proposed methods (3.1) and find out how to select parameters λ, ρ, a, b, c.

Applying the Taylor's series expansion of f about α, we get the following relations:

$$f(x_n) = f'(\alpha)\left[e_n + \theta_2 e_n^2 + \theta_3 e_n^3 + \theta_4 e_n^4 + O(e_n^5)\right], \tag{3.3}$$