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Efficient index reduction algorithm for large scale systems of differential algebraic equations

Xiaolin Qin[a,b,*], Juan Tang[a], Yong Feng[a], Bernhard Bachmann[c], Peter Fritzson[b]

[a] Chengdu Institute of Computer Applications, Chinese Academy of Sciences, Chengdu 610041, China
[b] Department of Computer and Information Science, Linköping University, Linköping SE-581 83, Sweden
[c] Department of Mathematics and Engineering, Bielefeld University of Applied Sciences, Bielefeld D-33609, Germany

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ABSTRACT

In many mathematical models of physical phenomenons and engineering fields, such as electrical circuits or mechanical multibody systems, which generate the differential algebraic equations (DAEs) systems naturally. In general, the feature of DAEs is a sparse large scale system of fully nonlinear and high index. To make use of its sparsity, this paper provides a simple and efficient algorithm for index reduction of large scale DAEs system. We exploit the shortest augmenting path algorithm for finding maximum value transversal (MVT) as well as block triangular forms (BTFs). We also present the extended signature matrix method with the block fixed point iteration and its complexity results. Furthermore, a range of nontrivial problems are demonstrated by our algorithm.

1. Introduction

The problem of differential algebraic equations (DAEs) system solving is fundamental in modeling many equation-based models of physical phenomenons and engineering fields, such as electric circuits [30,31], mechanical systems [28], spacecraft dynamics [24], chemical engineering [34], and many other areas. Generally, DAEs may be produced very large scale system of fully nonlinear and higher index in practice. However, most of the algorithms treat the low index case or consider solutions of linear systems [14,19,21,32]. The index is a notion used in the theory of DAEs for measuring the distance from a DAE to its related ordinary differential equation (ODE). It is well known that it is direct numerical computations difficult to solve a high index DAE. In particular, it may only solve some special classes of DAEs by the direct numerical solution [10,16,17].

Index reduction techniques can be used to remedy the drawback of direct numerical computation [3]. Pantelides' method [21] gives a systematic way to reduce high index systems of DAEs to lower index one, by selectively adding differentiated forms of the equations already present in the system. In [9], Ding et al. developed the weighted bipartite algorithm based on the minimally structurally singular subset, which is similar to the Pantelides' method. However, the algorithms can succeed yet not correctly in some instances [29]. Campbell's derivative array [4] needs to be computationally expensive especially for computing the singular value decomposition of the Jacobian of the derivative array equations using nonlinear singular least squares methods. Signature matrix method (also called Σ-method) [23] is based on solving an assignment problem, which can be formulated as an integer linear programming problem. In [23], Pryce proved that Σ-method is equivalent to the

* Corresponding author at: Chengdu Institute of Computer Applications, Chinese Academy of Sciences, Chengdu 610041, China. Tel.: +86 028 85249933; fax: +86 028 85229357.

E-mail addresses: qinxl@casit.ac.cn, qinxl2001@163.com (X. Qin), tangjuan0822@gmail.com (J. Tang).

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famous method of Pantelides' algorithm, and in particular computes the same structural index. However, the nice feature of Σ-method is a simple and straightforward method for analyzing the structure of DAEs of any order, not just first order.

In particular, large scale and high index DAEs with fully nonlinear systems are now routine that such models are built using interactive design systems based on the Modelica language [7,12]. In addition, the sparsity pattern of DAEs can arise in most actual applications [20,23,26]. In [11], Frenkel et al. gave a survey on appropriate matching algorithms based on the augmenting paths and push-relabel algorithm by translating Modelica models for large scale systems of DAEs. More recently, Wu et al. [35] generalized the Σ-method to the square and t-dominated partial differential equations (PDEs) systems. Pryce et al. [25] generalized the Σ-method for constructing a block triangular form (BTF) of the DAEs and exploiting to solve it efficiently in a block-wise manner. In [33], Tang et al. proposed the block fixed-point iteration with parameter method for DAEs based on its block upper triangular structure. However, the essential task is to solve the linear assignment problem for finding a maximum value transversal (MVT), which is a large part of the cost for index reduction of DAEs solving. Cao mentioned only in their work using Cao's *Matlab* implementation [8] of the shortest augmenting path algorithm of Jonker and Volgenant in [13]. We focus on solving in the large scale and high index cases in order to provide the shortest augmenting path algorithm for finding an MVT and an extended signature matrix method. The problem is also closely related to computing the block triangular form of a sparse matrix and linear assignment problems over integer.

Our approach is based on signature matrix method and modified Dijkstra's shortest path method. Our fundamental tool is the block triangularization of a sparse matrix; we exploit recent advances in linear assignment problem solving, which is equivalent to finding a maximum weight perfect matching in a bipartite graph of signature matrix in Σ-method, and we adapt the block fixed-point iteration with parameter for the canonical offsets techniques. Moreover, we give the new complexity analysis of our extended signature matrix method, which is superior to the latest results in Ref. [33]. Currently, we are working on the theoretical foundation and implementation of these methods on *Maple* platform. Another direction for future work is to exploit the fact that our algorithms are expressed in OpenModelica solvers.

The rest of this paper is organized as follows. The next section introduces our purpose and the shortest augmenting paths based algorithms, and presents an improved algorithm for the block triangularization for DAEs system. Section 3 describes the extended signature matrix method for the index reduction of large scale DAEs system and gives its complexity results. The following section shows our algorithm for an actual industrial example and some experimental results. The final section makes conclusions.

2. Preliminary results

2.1. Purpose

We consider an input DAEs system in n dependent variables $x_j = x_j(t)$ with t a scalar independent variable, of the general form

$$f_i(t, \text{the } x_j \text{ and derivatives of them}) = 0, \quad 1 \le i, j \le n. \tag{1}$$

The f_i are assumed suitably smooth, and the derivatives of x_j are arbitrary order. In general, signature matrix method is an effective preprocessing algorithm for the small and middle scale DAEs system. First, it needs to form the $n \times n$ signature matrix $\Sigma = (\sigma_{ij})$ of the DAEs, where

$$\sigma_{ij} = \begin{cases} \text{highest order of derivative to which the variable } x_j \text{ appears in equation } f_i, \\ \text{or } -\infty \text{ if the variable does not occur.} \end{cases}$$

Then, taken the analysis procedure of Σ as a linear assignment problem is to seek the offsets of the DAEs, that is, the number of differentiations of f_i. It can be formulated by the following primal problem:

$$\begin{aligned} \text{Maximize} \quad & z = \sum_{(i,j)\in S} \sigma_{ij}\xi_{ij}, \\ \text{subject to} \quad & \sum_{\{j|(i,j)\in S\}} \xi_{ij} = 1 \quad \forall\, i = 1, 2, \ldots, n, \\ & \sum_{\{i|(i,j)\in S\}} \xi_{ij} = 1 \quad \forall\, j = 1, 2, \ldots, n, \\ & \xi_{ij} \in \{0, 1\} \qquad \forall\, (i, j) \in S. \end{aligned} \tag{2}$$

Note that the state variable ξ_{ij} only be defined over the sparsity pattern of the problem

$$S = sparse(\Sigma) = \{(i, j)|\sigma_{ij} > -\infty\}. \tag{3}$$

It can be also defined on an undirected bipartite graph, in which case an assignment is a perfect matching. Given a bipartite graph $G(\Sigma) = (F, X, e)$, where F is the set vertices corresponding to the rows of Σ, X is the set vertices corresponding to the columns of Σ, and e is the set of edges corresponding to the non-negatively infinite in Σ, $|F| = |X| = n$, $|\cdot|$ denotes the cardinality of a set. In this paper, our goal is to handle large scale systems with n involving thousands and even more.

2.2. Block triangularization for DAEs system

As we have encountered with the increasingly large problems, an important preprocessing manipulation is the block triangularization of the system, which allows to split the overall system into subsystems which can be solved in the sequence