Applied Mathematics and Computation 277 (2016) 164–179

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Enabling local time stepping in the parallel implicit solution of reaction–diffusion equations via space-time finite elements on shallow tree meshes

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ARTICLE INFO

Keywords:
Local time stepping
Space-time adaptivity
Reaction-diffusion equations
Implicit solution schemes
Parallelization

ABSTRACT

For many applications, local time stepping offers an interesting and worthwhile alternative to the by now well established global time step control. In fact, local time stepping can allow for a highly detailed resolution of localized features of the solution with strongly reduced computational cost, when compared to global time step control. However local time stepping is not applicable in a straight-forward manner in the context of fully implicit time-discretizations.

Here, we present a method for the efficient parallel adaptive solution of (non-linear) partial differential equations, in particular reaction–diffusion equations, using spatially adapted time step sizes in the context of a fully implicit solution strategy. Our proposed method uses a discontinuous Galerkin method in-time approach within a full space-time approach. Moreover, it is designed from scratch for efficient parallel computation. We employ shallow tree-based mesh data structures in order to ensure a low memory footprint of the adaptive meshes. By solving the time-dependent partial differential equation on a $(d+1)$-dimensional non-conforming mesh, space-time adaptivity is naturally achieved. In combination with a discontinuous Galerkin method in-time the size of the arising systems can be precisely controlled.

We additionally introduce and discuss a stabilization scheme for space-time mortar element methods that also has a highly positive impact on the efficiency of preconditioning techniques for the arising systems of equations. We present results from extensive numerical experiments that address the question of convergence and efficiency, linear and non-linear solver performance, parallel scalability up to 2048 cores as well as accuracy for the linear heat equation and a real world, non-linear reaction–diffusion equation from the field of computational electrocardiology.

1. Introduction

Space-time adaptivity is often achieved by an appropriate combination of spatially adaptive techniques (e.g., adaptive finite element approximations on unstructured meshes) in combination with global time step control (see, for example,

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http://dx.doi.org/10.1016/j.amc.2015.12.017
0096-3003/

Lang [29] and the references therein). However in many application fields, for example in computational electrocardiology [39], global time step control is inefficient because the global time step size is forced to be small by strongly local features of the solution.

In such scenarios, local time stepping is an interesting alternative. In contrast to global time step control, local time stepping allows for using different time step sizes in different spatial regions. Local time stepping is a standard technique in structured adaptive mesh refinement codes for the solution of hyperbolic problems using explicit time integration, see, e.g., Berger and Colella [4]. However, local time stepping in combination with fully or semi-implicit time discretization schemes is a challenging task due to the global coupling inherent to the implicit approach.

In this article, we consider a space-time discretization on non-conforming meshes as an approach for implementing local time stepping in a (fully) implicit setting. Our discretization scheme is an extension of the work by Jamet [26] to the recently proposed shallow tree meshes [27,28], i.e. adaptive meshes based on local tree-data structures with a bounded number of levels. The advantage of the shallow tree mesh data structure is the low memory footprint which is of great relevant for space-time methods. Our method can be used in combination with conforming discretizations but also fits in the framework of mortar methods where weak continuity of the solution along non-matching mesh interfaces is enforced. In the context of the proposed space-time adaptive method we have to deal with the evaluation of boundary integrals over functions from mixed approximation spaces. Here, we additionally introduce a space-time transfer operator used in our discretization scheme, which greatly simplifies implementation and parallelization of this integral evaluation procedure. Moreover, we discuss and test a stabilization scheme for both conforming and non-conforming (mortar element) space-time discretization. The novel contribution of this article is the new combination of the discretization scheme by Jamet [26] with conforming or non-conforming discretization schemes on shallow tree meshes in a completely parallel setting. In addition to the newly suggested method, we present the results of extensive numerical experiments – addressing questions of convergence and efficiency, solver performance, parallel scalability as well as accuracy.

The article is organized as follows. In Section 3 we introduce the employed space-time discretization scheme on shallow tree meshes. A space-time transfer operator and stabilization term are proposed. Sections 4 and 5 contain extensive experiments for the linear heat equation and non-linear monodomain reaction–diffusion equation. Section 2 contains a discussion of related work in the literature. Finally, in Section 6 we discuss and summarize our results.

2. Related work

Local time stepping is a standard technique for the solution of hyperbolic equations using explicit time discretization schemes in combination with block-structured adaptive mesh refinement methods [4,5,17]. Gassner et al. [23] discuss local time stepping for discontinuous Galerkin discretizations based on explicit predictor-corrector schemes. Local time stepping in the context of wavelet-based adaptive resolution schemes using explicit time discretization schemes is discussed, for example, by Domingues et al. [19] and Bendahmane et al. [2,3]. Coquel et al. [10] discuss local time stepping for semi-implicit discretization schemes and adaptive resolution methods.

For a historical review of locally adaptive time stepping techniques we refer to Gander and Halpern [22].

Griebel and Oeltz [24] use space-time sparse grids for the solution of parabolic partial differential equations with continuous and discontinuous ansatz functions in time. By using sparse grids, the dimension of the ansatz space can be reduced from $O(N^{d+1})$ to $O(N^d)$, i.e., the order of the dimension of a stationary problem [25]. In contrast to classical sparse grids, the space-time sparse grids constructed by Griebel and Oeltz are not limited to tensor product spaces but can be used with an arbitrary multi-level basis in space. The authors discuss an adaptive discretization of parabolic equations with non-smooth solutions for which the regularity requirements of the sparse-grid approximation does not hold.

Yu [42] describes an implementation of local time stepping based on a multiplicative Schwarz domain decomposition method. In this method the finite element space is decomposed according to an overlapping decomposition of the domain Ω. Using the method of lines, the considered time-dependent partial differential equation is reduced to a coupled set of ordinary differential equations in the finite element space. These equations are solved in the interior of the local subdomains and the solutions in the subspaces are combined using a multiplicative Schwarz algorithm. Since the ordinary differential equations in different subdomains are solved independently (though in sequential order), different time steps or even different discretization schemes may be used in different domains. This approach has later been combined with a block-structured adaptive mesh refinement technique [43].

Tezduyar and Sathe [36] propose the enhanced-discretization space-time technique (EDSTT) to enable local time stepping for fluid dynamics and fluid structure interaction. They introduce two techniques. The first method, EDSTT-SM (single mesh), is based on a space-time conforming mesh that is refined towards the region of interest. The second method, EDSTT-MM (multi mesh), uses overlapping meshes of different resolution. In this method, the solution is locally written as a sum of contributions from ansatz spaces corresponding to the different meshes [37]. The authors present (1+1)- and (2+1)-dimensional results. Similar to this work, we propose to use a space-time discretization to implement local time stepping in an implicit setting. Both works are based on a discontinuous Galerkin approximation in time. In contrast to Tezduyar and Sathe we use non-overlapping, non-conforming meshes and standard finite element ansatz spaces for the discretization within a space-time slab. This ensures that our method can be used for solving a large class of partial differential equations. Whereas Tezduyar and Sathe consider a fixed spatial region where a smaller time step is used, in our approach the time step is adapted based on error estimates.