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Asymptotically polynomial solutions to difference equations of neutral type

Janusz Migda*

Faculty of Mathematics and Computer Science, Adam Mickiewicz University, Umultowska 87, 61-614 Poznań, Poland

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ABSTRACT

Asymptotic properties of solutions to difference equation of the form

$$\Delta^m(x_n + u_n x_{n+k}) = a_n f(n, x_{\sigma(n)}) + b_n$$

are studied. We give sufficient conditions under which all solutions, or all solutions with polynomial growth, or all nonoscillatory solutions are asymptotically polynomial. We use a new technique which allows us to control the degree of approximation.

1. Introduction

Let $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$ denote the set of positive integers, all integers and real numbers respectively. In this paper we assume that $m \in \mathbb{N}$ and consider difference equations of the form

$$\Delta^m(x_n + u_n x_{n+k}) = a_n f(n, x_{\sigma(n)}) + b_n \tag{E}$$

$$k \in \mathbb{Z}, \quad u_n, a_n, b_n \in \mathbb{R}, \quad f : \mathbb{N} \times \mathbb{R} \to \mathbb{R}, \quad \sigma : \mathbb{N} \to \mathbb{Z}.$$

Moreover, we assume

$$\sigma(n) \to \infty, \quad u_n \to c \in \mathbb{R}, \quad \text{and} \quad |c| \neq 1.$$

By a solution of (E) we mean a sequence $x : \mathbb{N} \to \mathbb{R}$ satisfying (E) for all large n.

In the series of papers [17–24] we present a new approach to the theory of asymptotic properties of solutions to difference equations. Usually the study of asymptotic properties of solutions is based on the equivalence relation defined as follows: two sequences x, y are called asymptotically equivalent if $x_n - y_n = \mathrm{o}(1)$. We replace $\mathrm{o}(1)$ by some other 'small' sequences, for example we use $\mathrm{o}(n^s)$ for certain $s \in (-\infty, 0]$.

Our approach is based on using the iterated remainder operator, the regional topology on the space of all real sequences and the 'regional' version of the Schauder fixed point theorem. The iterated remainder operator has been studied in [21]. In [22] the regional topology was introduced and its basic properties were considered. In our paper, similarly as in [24], we extend our approach to the case of equations of neutral type. In [24], using a new, 'regional', version of the Krasnoselsky fixed point theorem, we present some sufficient conditions for the existence of solutions with prescribed asymptotic behavior. In this paper we consider a problem which is in some sense inverse; we establish sufficient conditions under which all solutions, or all solutions with polynomial growth, or all nonoscillatory solutions are asymptotically polynomial.

* Tel.: +48 509744326.
E-mail address: migda@amu.edu.pl

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Asymptotic properties of solutions to neutral difference equations were investigated by many authors. These studies tend in several directions. For example, the papers [5,12,25,27], and [40] are devoted to the classification of solutions. Solutions with prescribed asymptotic behavior were studied in [8,10,14–16,24,32], and [33]. Oscillatory solutions were investigated in [1–4,13,34], and [39]. Asymptotically polynomial solutions were studied in [11,26,28,35–37].

Thandapani et al. in [36] establish conditions under which for any nonoscillatory solution x of the equation

$$\Delta^2(x_n + px_{n+k}) = f(n, x_{n+l}) \tag{1}$$

there exists a constant a such that

$$x_n = an + \mathrm{o}(n). \tag{2}$$

In [26], there are given conditions under which any nonoscillatory solution x of Eq. (1) has an asymptotic behavior

$$x_n = an + b + \mathrm{o}(1). \tag{3}$$

Migda, in [28], established conditions under which for any nonoscillatory solution x of (E) there exists a constant a such that

$$x_n = an^{m-1} + \mathrm{o}(n^{m-1}). \tag{4}$$

In Theorem 1, we extend these results in the following way. Let $s \in (-\infty, m-1]$ and let p be a nonnegative integer such that $s \le p \le m-1$. We establish conditions under which any solution, or any solution with polynomial growth, or any nonoscillatory solution x has an asymptotic behavior

$$x_n = a_{m-1}n^{m-1} + a_{m-2}n^{m-2} + \cdots + a_p n^p + \mathrm{o}(n^s) \tag{5}$$

for some fixed real $a_{m-1}, a_{m-2}, \dots, a_p$.

Asymptotically polynomial solutions to equations of neutral type were also studied in the 'continuous case'. But, to the best of our knowledge, there is not known a continuous counterpart of Theorem 1. In the paper [9], by Hasanbulli and Rogovchenko, for a class of mth order nonlinear neutral differential equations, sufficient conditions for all nonoscillatory solutions to satisfy

$$x(t) = at^{m-1} + \mathrm{o}(t^{m-1}) \tag{6}$$

are obtained. Condition (6) is a 'continuous' analogue of (4). For the history of this topic see [7] and [9].

Our main result, Theorem 1 and even its consequence, Corollary 4.2, significantly generalizes known results. The long proof of this theorem is divided into eight lemmas, five of them in Section 3 and three of them in Section 4. The idea of the proof is as follows. Let z be a sequence defined by

$$z_n = x_n + u_n x_{n+k}. \tag{7}$$

Using z we can write equation (E) in the form

$$\Delta^m z_n = a_n f(n, x_{\sigma(n)}) + b_n. \tag{8}$$

Let s be a real number such that $s \le m-1$. Assume that

$$\sum_{n=1}^{\infty} n^{m-1-s}|a_n| < \infty \qquad \text{and} \qquad \sum_{n=1}^{\infty} n^{m-1-s}|b_n| < \infty.$$

Using a Bihari type lemma and some additional assumptions, we show that (8) implies

$$\sum_{n=1}^{\infty} n^{m-1-s}|\Delta^m z_n| < \infty. \tag{9}$$

Next we use the result from [19], which states that if $\Delta^m z$ is asymptotically zero, then z is asymptotically polynomial. More precisely, we show that (9) implies

$$z_n = \varphi(n) + \mathrm{o}(n^s) \tag{10}$$

where φ is a polynomial sequence such that $\deg\varphi < m$. Finally, using our Lemma 3.5, we show that

$$x_n = \psi(n) + \mathrm{o}(n^s) \tag{11}$$

for certain polynomial sequence ψ such that $\deg\psi < m$.

The paper is organized as follows. In Section 2, we introduce notation and terminology. Section 3 is devoted to the proof of the fundamental Lemma 3.5. In Section 4, we obtain Theorem 1 and some consequences of this theorem. Section 5 contains a result analogous to Theorem 1, but we replace the spaces of asymptotically polynomial sequences by the spaces of regularly asymptotically polynomial sequences. More precise, we show, that if $s = q$ is a nonnegative integer, then (11) may be replaced by a stronger condition

$$x_n = \psi(n) + w_n, \qquad \Delta^k w_n = \mathrm{o}(n^{q-k}) \quad \text{for} \quad k = 0, 1, \dots, q.$$

In the paper we assume $m \in \mathbb{N}$, but in some lemmas, if m denotes the degree of a polynomial, we also allow $m = 0$ or $m = -1$.