Applied Mathematics and Computation 279 (2016) 43–61

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

The time finite element as a robust general scheme for solving nonlinear dynamic equations including chaotic systems

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ARTICLE INFO

Keywords:
Time finite element method
Chaotic system
Nonlinear dynamics

ABSTRACT

Schemes that can be proven to be unconditionally stable in the linear context can yield unstable solutions when used to solve nonlinear dynamical problems. Hence, the formulation of numerical strategies for nonlinear dynamical problems can be particularly challenging. In this work, we show that time finite element methods because of their inherent energy-momentum conserving property (in the case of linear and nonlinear elastodynamics), provide a robust time-stepping method for nonlinear dynamic equations (including chaotic systems). We also show that most of the existing schemes that are known to be robust for parabolic or hyperbolic problems can be derived within the time finite element framework; thus, the time finite element provides a unification of time-stepping schemes used in diverse disciplines. We demonstrate the robust performance of the time finite element method on several challenging examples from the literature where the solution behavior is known to be chaotic.

1. Introduction

Chaos is defined as the sensitivity to initial conditions of the solution to a nonlinear system of equations. If the numerical strategy that is used to solve the nonlinear system of equations is not robust enough, the 'chaos' that is seen in the solution can be more due to numerical instabilities in the scheme, rather than due to any inherent characteristic of the physical system being studied. Numerous examples of such 'numerical chaos' exist. For example, Xie and Steven [1] show that the use of the Newmark time-stepping scheme, which is one of the most popular and effective numerical integration schemes in structural dynamics, can lead to a chaotic response when applied to nonlinear dynamic equations. In fact, they show that the solution switches from stable to unstable when the time step is decreased!

In view of the above observations, the need for robust numerical schemes to solve nonlinear dynamic equations is of paramount importance, especially while investigating phenomena such as chaos, where any observed chaos in the solution should be purely due to physical (and not numerical) reasons. Several attempts have been made in the literature to develop such robust schemes. Xie and Steven [1] propose a numerical strategy that conserves energy exactly. However, this strategy is not derived within the time finite element framework, and can lead to large errors as we discuss in Section 4.5. Similarly, Bui [2] suggests a modification of the Newmark family that conserves energy exactly; however, the angular momentum

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http://dx.doi.org/10.1016/j.amc.2015.12.007
0096-3003/

is not conserved by this scheme. An extensive study and comparison of existing schemes has been carried out in [3,4][1]. The conclusion of the more recent reference of the two [4] is that, among existing schemes, the differential quadrature (DQ) scheme is perhaps the most accurate time-stepping strategy, and that the numerous other schemes considered such as Runge–Kutta, Newmark, α-method etc. can result in large errors (sometimes of the order of 100%). For conservative systems, the DQ scheme conserves energy only in an approximate sense, while, as already mentioned, the time finite element conserves energy exactly. Thus, from a stability viewpoint, the time finite element is superior to the DQ method. However, for highly nonlinear systems, from an accuracy perspective, one often needs a fine time step, in which case the DQ method also proves almost as reliable as the time finite element method. Thus, the time finite element method is to be considered as an alternative to the DQ method, with both methods having similar robustness and efficiency. One of the goals of this paper is to show that widely used time-stepping schemes such as the Crank–Nicolson method for linear parabolic problems, or the energy-momentum conserving scheme of Simo and Tarnow [7] for nonlinear elastodynamics (in an approximate sense) are special cases of the linear time finite element method. For small-sized problems, it may be preferable to use higher-order time finite elements, while for large-scale problems one may be constrained to use the *linear* one-step time finite element method, since otherwise the memory requirements would be prohibitive.

Generally, the time finite element method has been applied to conservative systems only [6,8–10]. In this work, we show (perhaps for the first time) that the time finite element method also yields very reliable results for chaotic systems (which can be conservative or nonconservative), by comparing the numerical results with analytical solutions where available or with other numerical solutions such as those obtained using the DQ method. Tang and Sun [10] have shown that time finite elements are related Runge–Kutta methods with infinitely many stages. We show in this work that the time finite element method is very closely related to the DQ method, and also that many existing time-stepping schemes for solving the semi-discrete equations derived from a continuum formulation are in fact special cases of the linear time finite element method. Thus, the time finite element strategy appears to be a robust and general method for the solution of linear and nonlinear transient problems.

2. Application of the time finite element method to chaotic systems

First, we briefly review the time finite element method for a system of first-order equations. In what follows, a superposed dot denotes a derivative with respect to time. We are interested in solving numerically the set of first-order differential equations

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}), \tag{1}$$

over the time interval $[0, T]$, subject to the initial conditions $\boldsymbol{x}(0) = \boldsymbol{x}_0$. We first partition the interval into a number of finite elements of size $t_\Delta = t_{n+1} - t_n$, where $0 < t_1 < \ldots < t_n < t_{n+1} < \ldots T$. We are interested in developing a time-stepping strategy over a typical time interval $[t_n, t_{n+1}]$, where given the initial conditions at the start of the interval $\boldsymbol{x}_n := \boldsymbol{x}(t_n)$, we are interested in finding $\boldsymbol{x}_{n+1} := \boldsymbol{x}(t_{n+1})$, which in turn will act as the initial condition while conducting the time-stepping on the next interval $[t_{n+1}, t_{n+2}]$. Thus, starting from the initial conditions at $t = 0$, we march forward in time until the instant T is reached.

Exactly as in the standard finite element method, we interpolate $\boldsymbol{x}$ and t using Lagrange interpolation functions as

$$t = \boldsymbol{N}\hat{\boldsymbol{t}}, \tag{2}$$

$$\boldsymbol{x} = \boldsymbol{N}_x\hat{\boldsymbol{x}}, \tag{3}$$

where $\boldsymbol{N}$ and $\boldsymbol{N}_x$ are the matrices of the Lagrange interpolation functions, and $\hat{\boldsymbol{t}}$ and $\hat{\boldsymbol{x}}$ are nodal values of the respective variables. For example, if we are using a quadratic time finite element, and if $\boldsymbol{x} := (x, y)$, then, with $N_1 := -\xi(1-\xi)/2$, $N_2 = 1 - \xi^2$, $N_3 = \xi(1+\xi)/2$, we have

$$\boldsymbol{N} = \begin{bmatrix} N_1 & N_2 & N_3 \end{bmatrix}, \quad \hat{\boldsymbol{t}} = \begin{bmatrix} t_n \\ t_m \\ t_{n+1} \end{bmatrix}, \tag{4a}$$

$$\boldsymbol{N}_x = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 \end{bmatrix}, \quad \hat{\boldsymbol{x}} = \begin{bmatrix} x_n \\ y_n \\ x_m \\ y_m \\ x_{n+1} \\ y_{n+1} \end{bmatrix}, \tag{4b}$$

where we typically take $t_m = (t_n + t_{n+1})/2$.

[1] Both these surveys seem to be unaware of the work of Greenspan [5] that conserves energy exactly for conservative systems, and which is a special case of the linear time finite element method [6].