Applied Mathematics and Computation 279 (2016) 154–169

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Boundedness and persistence of delay differential equations with mixed nonlinearity

Leonid Berezansky[a], Elena Braverman[b,*]

[a] Department of Mathematics, Ben-Gurion University of the Negev, Beer-Sheva 84105, Israel
[b] Department of Mathematics and Statstics, University of Calgary, 2500 University Drive N.W., Calgary, AB T2N 1N4, Canada

ARTICLE INFO

MSC:
34K25
34K60
92D25
34K23

Keywords:
Nonlinear delay differential equations
A global positive solution
Persistent
permanent and unbounded solutions
Population dynamics models
Mackey–Glass equation

ABSTRACT

For a nonlinear equation with several variable delays

$$\dot{x}(t) = \sum_{k=1}^{m} f_k(t, x(h_1(t)), \ldots, x(h_l(t))) - g(t, x(t)),$$

where the functions f_k increase in some variables and decrease in the others, we obtain conditions when a positive solution exists on $[0, \infty)$, as well as explore boundedness and persistence of solutions. Finally, we present sufficient conditions when a solution is unbounded. Examples include the Mackey–Glass equation with non-monotone feedback and two variable delays; its solutions can be neither persistent nor bounded, unlike the well studied case when these two delays coincide.

1. Introduction

Many mathematical models of population dynamics can be written in the form of a scalar equation

$$\dot{x}(t) = f(x(t-\tau)) - x(t), \tag{1.1}$$

where f is a nonnegative continuous function describing reproduction or recruitment, τ is a positive number describing delay. Usually these models have a unique positive equilibrium K, and there is a well-developed theory on the global stability of the positive equilibrium of (1.1). This theory was applied to many well-known models described by Eq. (1.1) such as Nicholson's blowflies delay equation and Mackey–Glass equations.

Eq. (1.1) can be extended to the case when both the delay and the intrinsic growth rate are variable

$$\dot{x}(t) = r(t)[f(x(h(t))) - x(t)], \tag{1.2}$$

where $h(t) \le t$ and $r(t) > 0$ are Lebesgue measurable. Global stability results for Eq. (1.2) with applications to population dynamics can be found in [15,16,18–21,23,24] and references therein, see also [7–10].

Another generalization of (1.1) is the model with several production terms and nonlinear mortality

$$\dot{x}(t) = \sum_{k=1}^{m} f_k(t, x(h_1(t)), \ldots, x(h_l(t))) - g(t, x(t)), \tag{1.3}$$

* Corresponding author. Tel.: +1 403 220 3956; fax: +1 403 282 5150.
E-mail address: maelena@math.ucalgary.ca, maelena@ucalgary.ca (E. Braverman).

http://dx.doi.org/10.1016/j.amc.2016.01.015

where f_k, g are nonnegative continuous functions. This equation with some applications was studied, for example, in [3,5,8,17,25].

For all mentioned above equations usual assumptions are the following: the function f_k is either monotone or unimodal, $g(t, u)$ is monotone increasing in u, there is only one delay involved in f_k, and a positive equilibrium is unique. However, it is possible to consider more general models, for example, the modified Nicholson equation

$$\dot{x}(t) = \sum_{k=1}^{m} a_k(t)x(h_k(t))e^{-\lambda_k x(g_k(t))} - b(t)x(t), \quad t \ge 0, \tag{1.4}$$

and the modified Mackey–Glass type equation

$$\dot{x}(t) = \sum_{k=1}^{m} \frac{a_k(t)x(h_k(t))}{1+x^{n_k}(p_k(t))} - \left(b(t) - \frac{c(t)}{1+x^n(t)}\right)x(t), \quad t \ge 0. \tag{1.5}$$

There are also many generalizations of Eqs. (1.1)–(1.5) to the case of distributed delays and integro-differential equations [6,11,12,22,26].

Let us illustrate the idea that the presence of several delays instead of one delay can create a new type of dynamics. As Example 1.1 illustrates, an equation which was stable for coinciding delays can become unstable, once the two delays are different.

Example 1.1. Consider the modified Mackey–Glass equation with two delays

$$\dot{x}(t) = \frac{2x(h(t))}{1+x^2(g(t))} - x(t), \quad t \ge 0. \tag{1.6}$$

The unique positive equilibrium is $x = 1$, the function $f(x) = 2x/(1+x^2)$ is increasing on [0, 1], so any positive solution of the equation

$$\dot{x}(t) = \frac{2x(h(t))}{1+x^2(h(t))} - x(t), \quad t \ge 0 \tag{1.7}$$

satisfies $\lim_{t\to\infty} x(t) = 1$, see, for example, [10,12]. Consider (1.6) with piecewise constant arguments $h(\cdot)$, $g(\cdot)$. Denote $a = \ln(59/24) \approx 0.8994836$ and $b = \ln(134/15) \approx 2.1897896$ and let

$$\varphi(t) = 6.4 - 5.9e^{-(t+a+b)}, \quad t \in [-a-b, -b], \quad \varphi(t) = \frac{1}{17} + \frac{67}{17}e^{-(t+b)}, \quad t \in [-b, 0],$$

then $\varphi(-a-b) = 0.5$, $\varphi(-b) = 4$, $\varphi(0) = \frac{1}{17} + \frac{67}{17}\frac{15}{134} = 0.5$. Assume for $n = 0, 1, 2, \dots$

$$h(t) = \begin{cases} \left[\frac{t}{a+b}\right] - b, & t \in [n(a+b), n(a+b)+a), \\ \left[\frac{t}{a+b}\right] - a - b, & t \in [n(a+b)+a, (n+1)(a+b)), \end{cases}$$

where $[t]$ is the integer part of t,

$$g(t) = \begin{cases} \left[\frac{t}{a+b}\right] - a - b, & t \in [n(a+b), n(a+b)+a), \\ \left[\frac{t}{a+b}\right] - b, & t \in [n(a+b)+a, (n+1)(a+b)). \end{cases}$$

Then the solution is $(a+b)$-periodic, the equation is $\dot{x}(t) = \frac{32}{5} - x(t)$ on $[n(a+b), n(a+b)+a)$, $x(n(a+b)) = \frac{1}{2}$ and $\dot{x}(t) = \frac{1}{17} - x(t)$ on $[n(a+b)+a, (n+1)(a+b))$, $x(n(a+b)+a) = 4$. Thus, with two delays, the equilibrium $K = 1$ of Eq. (1.6) is not globally asymptotically stable, unlike (1.7).

As Example 1.1 illustrates, an equation which was stable for the coinciding delays can have oscillating solutions with a constant amplitude which do not tend to the positive equilibrium. According to Example 5.8, two different delays can lead not only to sustainable oscillations but also to unbounded solutions.

The purpose of the present paper is to consider a general nonlinear delay equation which includes (1.4), (1.5) as particular cases and study the following properties of these equations: existence and uniqueness of a positive global solution, persistence, permanence, as well as existence of unbounded solutions. To the best of our knowledge, equations with such mixed types of nonlinearities have not been studied before.

Compared to most of the previous publications, we consider two modifications: the production function is a sum of several functions, and each f_k involves several delays. The situation when several (sometimes incomparable) delays are included, is quite common, for example, transmission and translation delays in gene regulatory systems. Motivated by this, we apply the general results to some well-known population dynamics equations.

The paper is organized as follows. After introducing some relevant assumptions and definitions in Section 2, we justify existence of a global positive solution in Section 3. Section 4 deals with sufficient conditions when all positive solutions are bounded. In Section 5, we investigate persistence of solutions and also consider their permanence. Section 6 explores positive unbounded solutions, and Section 7 involves brief discussion.