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An analysis of a Khattri's 4th order family of methods

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ABSTRACT

In this paper we analyze an optimal fourth-order family of methods suggested by Khattri and Babajee, (2013). We analyze the family using the information on the extraneous fixed points. Two measures of closeness to the imaginary axis of the set of extraneous points are considered and applied to the members of the family to find its best performer. The results are compared to three best members of King's family of methods.

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1. Introduction

"Calculating zeros of a scalar function f ranks among the most significant problems in the theory and practice not only of applied mathematics, but also of many branches of engineering sciences, physics, computer science, finance, to mention only some fields" [2]. For example, to minimize a function $F(x)$ one has to find the points where the derivative vanishes, i.e. $F'(x) = 0$. There are many algorithms for the solution of nonlinear equations, see e.g. Traub [3], Neta [4] and the recent book by Petković et al. [2]. The methods can be classified as one step and multistep. One step methods are of the form

$$x_{n+1} = \phi(x_n).$$

The iteration function ϕ depends on the method used. For example, Newton's method is given by

$$x_{n+1} = \phi(x_n) = x_n - \frac{f(x_n)}{f'(x_n)}. \tag{1}$$

Some one point methods allow the use of one or more previously found points, in such cases we have a one step method with memory. For example, the secant method uses one previous point and is given by

$$x_{n+1} = x_n - \frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})} f(x_n).$$

In order to increase the order of a one step method, one requires higher derivatives. For example, Halley's method is of third order and uses a second derivative [5]. In many cases the function is not smooth enough or the higher derivatives are too complicated. Another way to increase the order is by using multistep. The recent book by Petković et al. [2] is dedicated to multistep methods. A trivial example of a multistep method is a combination of two Newton steps, i.e.

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \tag{2}$$

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$$x_{n+1} = y_n - \frac{f(y_n)}{f'(y_n)}. \tag{2}$$

Of course this is too expensive. The cost of a method is defined by the number (ℓ) of function-evaluations per step. The method (2) requires four function-evaluations (including derivatives). The efficiency of a method is defined by

$$I = p^{1/\ell},$$

where p is the order of the method. Clearly one strives to find the most efficient methods. To this end, Kung and Traub [6] introduced the idea of optimality. They conjectured that a method using ℓ evaluations is optimal if the order is $2^{\ell-1}$. This conjecture was proved by Woźniakowski [7] in the case of Hermitian information. Kung and Traub have developed optimal multistep methods of increasing order. Newton's method (1) is optimal of order 2. King [8] has developed an optimal fourth order family of methods depending on a parameter β

$$\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f'(x_n)}, \\ x_{n+1} &= y_n - \frac{f(y_n)}{f'(x_n)} \frac{f(x_n) + \beta f(y_n)}{f_n + (\beta - 2) f(y_n)}, \end{aligned} \tag{3}$$

Neta [9] has developed a family of sixth order methods based on King's method (3). Also Neta [10] has developed optimal eighth and sixteenth order methods. Wang and Liu [11] and Thukral and Petković [12] have developed optimal eighth order methods. Khattri and Babajee [1] has developed the following optimal fourth order 3 parameter family of methods

$$\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f'(x_n) + \frac{\alpha\beta}{2} f(x_n)^m}, \\ x_{n+1} &= y_n - \frac{f(x_n) f(y_n)}{f(x_n) - 2f(y_n)} \left[\frac{\alpha}{f'(x_n) + \beta f(x_n)^m} - \frac{\alpha - 1}{f'(x_n) + \eta f(y_n)} \right]. \end{aligned} \tag{4}$$

There are a number of ways to compare various techniques proposed for solving nonlinear equations. Comparisons of the various algorithms are based on the number of iterations required for convergence, number of function evaluations, and/or amount of CPU time. "The primary flaw in this type of comparison is that the starting point, although it may have been chosen at random, represents only one of an infinite number of other choices" [13]. In recent years the Basin of Attraction method was introduced to visually comprehend how an algorithm behaves as a function of the various starting points. The first comparative study using basin of attraction, to the best of our knowledge, is by Vrscay and Gilbert [14]. They analyzed Schröder and König rational iteration functions. Other work was done by Stewart [15], Kalantari and Jin [16], Amat et al. [17–20], Chicharro et al. [21], Chun et al. [22,23], Cordero et al. [24], Neta et al. [25–27], Magreñán [28], Magreñán et al. [29], and Scott et al. [13]. There are also similar results for methods to find roots with multiplicity, see e.g. [30,31] and [32].

In this paper we analyze a family of optimal fourth order methods (4). We will examine the family and show how to choose the parameters involved in the family similar to Chun et al. [33].

2. Extraneous fixed points

In solving a nonlinear equation iteratively we are looking for fixed points which are zeros of the given nonlinear function. Many multipoint iterative methods have fixed points that are not zeros of the function of interest. Thus, it is necessary to investigate the number of extraneous fixed points, their location and their properties. In order to find the extraneous fixed points, we rewrite the family of methods in the form

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} H_f(x_n, y_n), \tag{5}$$

where the function H_f for method (4) is given by

$$H_f(x_n, y_n) = \frac{f'(x_n)}{f'(x_n) + \frac{\alpha\beta}{2} f(x_n)^m} + \frac{f'(x_n) f(y_n)}{f(x_n) - 2f(y_n)} \left[\frac{\alpha}{f'(x_n) + \beta f(x_n)^m} - \frac{\alpha - 1}{f'(x_n) + \eta f(y_n)} \right]. \tag{6}$$

Clearly, if x_n is the root then from (5) we have $x_{n+1} = x_n$ and the iterative process converged. But we can have $x_{n+1} = x_n$ even if x_n is not the root but $H_f(x_n, y_n) = 0$. Those latter points are called extraneous fixed points. It is best to have the extraneous fixed points on the imaginary axis or close to it. For example, in the case of King's method (3) we found that the best performance is when the parameter $\beta = 3 - 2\sqrt{2}$ or $\beta = 0$ since then the extraneous fixed points are closest to the imaginary axis.

We have searched the parameter space (α, β, η) in the case of $m = 1$ and found that the extraneous fixed points are not on the imaginary axis except in the case that any two of the parameters are zero, which is Ostrwoski's fourth order method [3]. As it can be seen in the next section, the cases of m greater than 1 (i.e. methods KB1 and KB2) gave worse performance than $m = 1$. We have tried to get several measures of closeness to the imaginary axis and experimented with those members from the parameter space.