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hp-discontinuous Galerkin method based on local higher order reconstruction ☆

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ABSTRACT

We present a new adaptive higher-order finite element method (*hp*-FEM) for the solution of boundary value problems formulated in terms of partial differential equations (PDEs). The method does not use any information about the problem to be solved which makes it robust and equation-independent. It employs a higher-order reconstruction scheme over local element patches which makes it faster and easier to parallelize compared to *hp*-adaptive methods that are based on the solution of a reference problem on a globally *hp*-refined mesh. The method can be used for the solution of linear as well as nonlinear problems discretized by conforming or non-conforming finite element methods, and it can be combined with arbitrary a posteriori error estimators. The performance of the method is demonstrated by several examples carried out by the discontinuous Galerkin method.

1. Introduction

Adaptive methods are an efficient tool for the numerical solution of PDEs. Automatic mesh refinement or, more generally, an enhancement of the functional space where the approximate solution is sought, can significantly reduce the computational cost. A prominent place among adaptive methods has the *hp*-FEM which leads to unconditional exponential convergence [1–4].

Various approaches to automatic adaptivity include refining an element without increasing its polynomial degree (*h*-refinement), increasing the polynomial degree of an element without spatial subdivision (*p*-refinement), and performing refinements that combine spatial splitting of an element with various distributions of the polynomial degrees in subelements (genuine *hp*-refinement [5,6]).

To achieve exponential convergence, large higher-order elements must be used where the solution is smooth, and at the same time small low-order elements must be used where the solution exhibits non-smooth features such as singularities or internal/boundary layers. Therefore it would seem that looking at the smoothness of the solution is the best way to design a *hp*-adaptive method. However, *hp*-adaptive strategies based on smoothness estimation usually can only decide between *h*- and *p*-refinements because they do not have enough information to select optimal genuine *hp*-refinements [7–9].

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To take full advantage of genuine *hp*-refinements, one has to obtain more information about the error – not only as an error estimate in the form of a number per element, but about its shape as a function that is defined inside an element. This can be done by solving a reference problem on a globally *hp*-refined mesh [5,6,10]. It leads to superior convergence rates in terms of degrees of freedom, but the reference problem tends to be huge and therefore the convergence is rather slow in terms of computing time, see [11] where a deep comparison of different *hp*-strategies were compared.

In this paper we propose a novel method for automatic *hp*-adaptivity that is guided by the approximation error, but it removes the tedious computation of the global reference solution. Instead, it calculates a more accurate approximation on local element patches constructed for each element separately. With the aid of a weighted least square reconstruction, we construct piecewise polynomial function which approximate the exact solution. The presented *hp*-adaptive strategy is based on comparing the higher-order reconstruction with the approximate solution u_h and its L^2-projections to lower-degrees polynomial spaces. Then, for each candidate, we directly estimate the number of degrees of freedom necessary to achieve a local tolerance for each element and propose a new (better) polynomial approximation degree and a new (better) element size. Consequently, a new triangular grid is constructed.

Although we originally developed this technique for the discontinuous Galerkin method, it can be simply modified to other types of finite element approximations including conforming finite elements, mixed finite elements, etc. Moreover, since this approach is based on the reconstruction of the approximate solution, we can employ it for arbitrary (linear as well as non-linear) boundary value problems. Finally, its extension to 3D problems is straightforward.

The outline of the paper is as follows: In Section 2, we introduce the governing equations and their discretization by the discontinuous Galerkin method. In Section 3, we introduce the higher-order reconstruction technique and formulate the saturation assumption. Its validity is numerically demonstrated in Section 4. The main novelty of this paper is presented in Section 5, where we present the *hp*-adaptive strategy. In Section 6, we present three numerical examples that demonstrate the performance and robustness of the proposed method.

2. Problem description

2.1. Governing equations

We consider the nonlinear *convection–diffusion problem*

$$\nabla \cdot \boldsymbol{f}(u) - \nabla \cdot (\boldsymbol{K}(u)\nabla u) = g(x), \tag{1a}$$

$$u|_{\partial\Omega_D} = u_D, \tag{1b}$$

$$\boldsymbol{K}(u)\frac{\partial u}{\partial \boldsymbol{n}}\Big|_{\partial\Omega_N} = g_N, \tag{1c}$$

where $u : \Omega \to \mathbb{R}$ is an unknown scalar function defined on $\Omega \in \mathbb{R}^2$. We assume that Ω is polygonal for simplicity. Moreover, $\boldsymbol{f}(u) = (f_1(u), f_2(u)) : \mathbb{R} \to \mathbb{R}^2$ and $\boldsymbol{K}(u) = \{K_{ij}(u)\}_{i,j=1}^2 : \mathbb{R} \to \mathbb{R}^{2\times 2}$ are nonlinear functions of their arguments, $\boldsymbol{n}$ is the unit outer normal to $\partial\Omega$ and $\emptyset \neq \partial\Omega_D \cup \partial\Omega_N = \partial\Omega$ are disjoint parts of the boundary of Ω. Symbols ∇ and $\nabla\cdot$ mean the gradient and divergence operators, respectively.

We assume that $f_s \in C^1(\mathbb{R}),\quad f_s(0) = 0,\ s = 1, 2,$ $\boldsymbol{K}$ is bounded and positively definite, $g \in L^2(\Omega)$, u_D is the trace of some $u^* \in H^1(\Omega) \cap L^\infty(\Omega)$ on $\partial\Omega_D$ and $g_N \in L^2(\partial\Omega_N)$. We use the standard notation for function spaces (see, e. g., [12]): $L^p(\Omega)$ denote the Lebesgue spaces, $W^{k,p}(\Omega)$, $H^k(\Omega) = W^{k,2}(\Omega)$ are the Sobolev spaces and $P^k(M)$ denotes the space of polynomial functions of degree $\le k$ defined on the domain $M \subset \mathbb{R}^2$. Let us note that any function from $P^k(M)$ can be interpreted as a polynomial function defined on $\mathbb{R}^2$ restricted to M. By $\phi|_M$ we denote the restriction of a function ϕ on M.

In order to introduce the weak solution, we define the spaces

$$V := \{v;\ v \in H^1(\Omega),\quad v|_{\partial\Omega_D} = 0\},\quad W := \{v;\ v \in H^1(\Omega),\quad v - u^* \in V\}. \tag{2}$$

We say that function u is the *weak solution* of (1), if the following conditions are satisfied

$$u \in W \cap L^\infty(\Omega), \tag{3a}$$

$$\int_\Omega [\nabla \cdot \boldsymbol{f}(u)\, v + (\boldsymbol{K}(u)\nabla u) \cdot \nabla v]\, \mathrm{d}x = \int_\Omega g v\, \mathrm{d}x + \int_{\partial\Omega_N} g_N\, v\, \mathrm{d}S \quad \forall v \in V. \tag{3b}$$

The assumption $u \in L^\infty(\Omega)$ in (3) guarantees the boundedness of functions $\boldsymbol{f}(u)$ and $\boldsymbol{K}(u)$ and therefore the existence of the integrals in (3a). This assumption can be weakened if functions $\boldsymbol{f}(u)$ and $\boldsymbol{K}(u)$ satisfy some growth conditions.

2.2. Discretization of the problem

Let $\mathscr{T}_h$ $(h > 0)$ be a partition of the closure $\overline{\Omega}$ of the domain Ω into a finite number of triangles K with mutually disjoint interiors. We call $\mathscr{T}_h = \{K\}_{K\in\mathscr{T}_h}$ a *triangulation* of Ω and for simplicity, we assume that $\mathscr{T}_h$ satisfies the conforming properties from the finite element method, see, e.g, [13]. The diameter of $K \in \mathscr{T}_h$ is denoted by h_K.