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Fokker–Planck equations for stochastic dynamical systems with symmetric Lévy motions

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ABSTRACT

The Fokker–Planck equations for stochastic dynamical systems, with non-Gaussian α-stable symmetric Lévy motions, have a nonlocal or fractional Laplacian term. This nonlocality is the manifestation of the effect of non-Gaussian fluctuations. Taking advantage of the Toeplitz matrix structure of the time-space discretization, a fast and accurate numerical algorithm is proposed to simulate the nonlocal Fokker–Planck equations on either a bounded or infinite domain. Under a specified condition, the scheme is shown to satisfy a discrete maximum principle and to be convergent. It is validated against a known exact solution and the numerical solutions obtained by using other methods. The numerical results for two prototypical stochastic systems, the Ornstein–Uhlenbeck system and the double-well system are shown.

1. Introduction

The Fokker–Planck (FP) equations for stochastic dynamical systems [1–3] describe the time evolution for the probability density of solution paths. When the noise in a system is Gaussian (i.e., in terms of Brownian motion), the corresponding Fokker–Planck equation is a differential equation. The Fokker–Planck equations are widely used to investigate stochastic dynamics in physical, chemical and biological systems. However, the noise is often non-Gaussian [4–9], e.g., in terms of α-stable Lévy motions. The corresponding Fokker–Planck equation has an integral term and is a differential–integral equation. In fact, this extra term is an integral over the whole state space and thus represents a nonlocal effect caused by the non-Gaussianity of the noise.

A Brownian motion is a Gaussian stochastic process, characterized by its mean or drift vector (taken to be zero for convenience), and a diffusion or covariance matrix. A Lévy process L_t on $\mathbb{R}^n$ is a non-Gaussian stochastic process. It is characterized by a drift vector $b \in \mathbb{R}^n$ (taken to be zero for convenience), a Gaussian covariance matrix A, and a non-negative Borel measure ν, defined on $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$ and concentrated on $\mathbb{R}^n \setminus \{0\}$. The measure ν satisfies the condition $\int_{\mathbb{R}^n\setminus\{0\}} (y^2 \wedge 1)\, \nu(dy) < \infty$, where $a \wedge b = \min\{a, b\}$, or equivalently $\int_{\mathbb{R}^n\setminus\{0\}} \frac{y^2}{1+y^2}\, \nu(dy) < \infty$. This measure ν is the so called Lévy jump measure of the Lévy process L_t. We also call (b, A, ν) the *generating triplet* [10,11].

In this paper, we consider stochastic differential equations (SDEs) with a special class of Lévy processes, the α-stable symmetric Lévy motions. The corresponding Fokker–Planck equations (for the evolution of the probability density function

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of the solution) contain a nonlocal term, i.e., the fractional Laplacian term, which quantifies the non-Gaussian effect. In theoretical physics, the SDEs under consideration are usually referred to as the Langevin equations with Lévy noise and the associated Fokker–Planck equations are named as fractional Fokker–Planck equations (FFPE) for Lévy flights (e.g. [12,13]). Except for some simple systems in the entire space $\mathbb{R}^n$ [14], it is hardly possible to have analytical solutions for these nonlocal Fokker–Planck equations especially for finite domains. We thus consider numerical simulation for these nonlocal equations, either on a bounded domain or a unbounded domain. Due to the nonlocality, however, the 'boundary' condition needs to be prescribed on the entire exterior domain. A recently developed nonlocal vector calculus has provided sufficient conditions for the well-posedness of the initial-boundary value problems on this type of nonlocal diffusion equations with volume constraints. See the review paper [15].

A few authors have considered numerical simulations of reaction–diffusion type partial differential equations with a (formal) fractional Laplacian operator. They impose a boundary condition only on the boundary of a bounded set (not on the entire exterior set as we do here); see [16], [[17], Eqs. (10) and (11)], [[18], Section 3], [19], [[20], Eq. (2)] and references therein. The domain of definition of the fractional Laplacian operator in these papers consists of certain functions with prescribed values on the boundary. However, the domain of definition of the fractional Laplacian operator in our present paper consists of certain functions with prescribed values on the entire exterior domain. Thus, the fractional Laplacian operator in the present paper is different from that in [16–20], as the domains of definition are different. Note that the exterior boundary condition is required for understanding probability density evolution. Recently, discontinuous and continuous Galerkin methods have been developed for the volume-constrained nonlocal diffusion problems and their error analysis is given in [15] and the references therein. To our knowledge, the work [21] is closely related to our work, where the authors of [21] have shown the well-posedness, the maximum principle, conservation and dispersion relations for a class of nonlocal convection–diffusion equations with volume constraints. A finite difference scheme is also presented in [21], which maintains the maximum principle when suitable conditions are met. Compared with this paper, the difference is that the jump processes considered in [21] are nonsymmetric and of finite-range.

Recently, there are numerous works on numerical methods for equations containing fractional Laplacian or fractional derivatives, e.g., [22–31]. These work are different from ours in the equations considered and/or the numerical methods used. In this work, we develop an explicit finite difference method for the FPE derived directly from the SDE, and the method is shown to be convergent both theoretically and numerically.

This paper is organized as follows. In Section 2, we present the nonlocal Fokker–Planck equation for a SDE with α-stable symmetric Lévy motion and then devise a numerical discretization scheme for both bounded and infinite domains. For simplicity, we consider scalar SDEs. The description and analysis of the numerical schemes are presented in Section 3 and Section 4, respectively. The numerical experiments are conducted in Section 5. The paper ends with some discussions in Section 6.

2. Fokker–Planck equations for SDEs with Lévy motions

Consider a scalar SDE

$$\mathrm{d}X_t = f(X_t)\,\mathrm{d}t + \mathrm{d}L_t, \quad X_0 = x_0, \tag{1}$$

where f is a given deterministic vector field, the scalar Lévy process L_t has the generating triplet $(0, d, \varepsilon\nu_\alpha)$, with diffusion constant $d \geq 0$ and the α-stable symmetric jump measure

$$\nu_\alpha(\mathrm{d}y) = C_\alpha |y|^{-(1+\alpha)}\,\mathrm{d}y, \quad \text{where} \quad C_\alpha = \frac{\alpha}{2^{1-\alpha}\sqrt{\pi}} \frac{\Gamma(\frac{1+\alpha}{2})}{\Gamma(1-\frac{\alpha}{2})}.$$

The non-Gaussianity index $\alpha \in (0, 2)$ and the intensity constant is ε. A symmetric 2-stable($\alpha = 2$) process is simply a Brownian motion. For more information on α-stable Lévy motions, see [32–34].

Some sufficient conditions under which the SDE (1) has unique pathwise solutions (e.g., similar to pointwise solutions for deterministic ordinary differential equations) or unique martingale solutions (e.g., a kind of 'weak' solutions which are appropriate for dealing with probability density and other statistical properties of solutions) have been considered in [35,36]. Under such conditions, the corresponding probability density for the solution exists. In this work, we assume that the probability density exists and has continuous second-order spatial derivatives in the interior of the domain and continuous first-order derivative in time.

The FP equation for the distribution of the conditional probability density $p(x, t) = \mathbb{P}(X_t = x | X_0 = x_0)$, i.e., the probability of the process X_t has value x at time t given it had value x_0 at time 0, is given as [1,10,32,34,37]

$$\partial_t p = -\partial_x (f(x)\ p) + \frac{1}{2} d \partial_{xx} p + \varepsilon[-(-\Delta)^{\frac{\alpha}{2}}]^* p. \tag{2}$$

The fractional Laplacian operator is defined by

$$-(-\Delta)^{\frac{\alpha}{2}} p = \int_{\mathbb{R}\setminus\{0\}} [p(x+y, t) - p(x, t) - I_{\{|y|<1\}}\ y\ \partial_x p(x, t)]\ \nu_\alpha(\mathrm{d}y). \tag{3}$$

There are many equivalent definitions of the fractional Laplacian operator. For instance, in the entire space $\mathbb{R}$, it is identical to the Riesz operator (e.g. [12]) because the Fourier transform $\mathcal{F}$ of the operator satisfies $\mathcal{F}[-(-\Delta)^{\frac{\alpha}{2}} p](\xi) = -|\xi|^\alpha \mathcal{F}[p](\xi)$.