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Oscillation of third-order nonlinear damped delay differential equations

Martin Bohner[a,*], Said R. Grace[b], Ilgin Sağer[a], Ercan Tunç[c]

[a] Missouri University of Science and Technology, Department of Mathematics and Statistics, Rolla, MO 65409-0020, USA
[b] Cairo University, Department of Engineering Mathematics, Faculty of Engineering, Orman, Giza 12221, Egypt
[c] Gaziosmanpaşa University, Department of Mathematics, Faculty of Arts and Sciences, Tokat 60240, Turkey

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ABSTRACT

This paper is concerned with the oscillation of certain third-order nonlinear delay differential equations with damping. We give new characterizations of oscillation of the third-order equation in terms of oscillation of a related, well-studied, second-order linear differential equation without damping. We also establish new oscillation results for the third-order equation by using the integral averaging technique due to Philos. Numerous examples are given throughout.

1. Introduction

In this article, we consider nonlinear third-order functional differential equations of the form

$$(r_2(r_1(y')^\alpha)')'(t) + p(t)(y'(t))^\alpha + q(t)f(y(g(t))) = 0, \quad t \geq t_0 > 0, \tag{1.1}$$

where $\alpha \geq 1$ is the ratio of positive odd integers. We assume that

(i) $r_1, r_2, p, q \in C(I, \mathbb{R}^+)$, where $I = [t_0, \infty)$ and $\mathbb{R}^+ = (0, \infty)$;
(ii) $g \in C^1(I, \mathbb{R})$, $g'(t) \geq 0$, and $g(t) \to \infty$ as $t \to \infty$;
(iii) $f \in C(\mathbb{R}, \mathbb{R})$ such that $xf(x) > 0$ for $x \neq 0$ and $f(x)/x^\beta \geq k > 0$, where β is a ratio of positive odd integers.

A function y is called a solution of (1.1) if

$$y, r_1(y')^\alpha, r_2(r_1(y')^\alpha)' \in C^1([t_y, \infty), \mathbb{R})$$

and y satisfies (1.1) on $[t_y, \infty)$ for some $t_y \geq t_0$.

We restrict our attention to those solutions of (1.1) which exist on $[t_0, \infty)$ and satisfy the condition $\sup\{|y(t)| : t_1 \leq t < \infty\} > 0$ for $t_1 \in I$. Such a solution is called *oscillatory* if it has arbitrarily large zeros, otherwise it is called *nonoscillatory*. Eq. (1.1) is said to be oscillatory if all of its solutions are oscillatory.

Determining oscillation criteria for particular second-order differential equations has received a great deal of attention in the last few years. Compared to second-order differential equations, the study of oscillation and asymptotic behavior of third-order differential equations has received considerably less attention in the literature. For some classical and recent

* Corresponding author. Tel.: +15733414129.
E-mail addresses: bohner@mst.edu (M. Bohner), saidgrace@yahoo.com (S.R. Grace), ilginsager81@gmail.com (I. Sağer), ercantunc72@yahoo.com (E. Tunç).

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results on third-order equations, the reader can refer to [1–4,7–9,15,16,20] and the references contained therein. It is interesting to note that there are third-order delay differential equations which have only oscillatory solutions or have both oscillatory and nonoscillatory solutions. For example,

$$y'''(t) + 2y'(t) - y\left(t - \frac{3\pi}{2}\right) = 0$$

admits the oscillatory solution $y_1(t) = \sin t$ and a nonoscillatory solution $y_2(t) = e^{\lambda t}$, where $\lambda > 0$ is such that $\lambda^3 + 2\lambda = e^{-3\pi/2}$ (see [14, p. 6]). On the other hand, all solutions of

$$y'''(t) + y(t - \tau) = 0, \quad \tau > 0$$

are oscillatory if and only if $\tau e > 3$ (see [13]). But the corresponding ordinary differential equation

$$y'''(t) + y(t) = 0$$

admits the nonoscillatory solution $y_1(t) = e^{-t}$ and oscillatory solutions $y_2(t) = e^{t/2}\sin(\sqrt{3}t/2)$ and $y_3(t) = e^{t/2}\cos(\sqrt{3}t/2)$.

In the literature, there are some papers and books (see, e.g., [5,6,11,17,19,21,22] and the references therein) which deal with the oscillatory and asymptotic behavior of solutions of functional differential equations. In [21], the authors used a generalized Riccati transformation and an integral averaging technique in order to establish some sufficient conditions which ensure that any solution of (1.1) oscillates or converges to zero. The purpose of this paper is to improve and unify the results in [21] and present some new sufficient conditions which ensure that any solution of (1.1) oscillates when the related second-order linear ordinary differential equation without delay

$$(r_2 z')'(t) + \left(\frac{p(t)}{r_1(t)}\right) z(t) = 0 \tag{1.2}$$

is nonoscillatory or oscillatory. We also apply our results to equations of the form

$$a_3(t)y'''(t) + a_2(t)y''(t) + a_1(t)y'(t) + q^*(t)f(y(g(t))) = 0, \tag{1.3}$$

where $a_1, a_2, a_3, q^* \in \mathrm{C}(I, \mathbb{R}^+)$.

2. Auxiliary results

For the sake of brevity, we define

$$L_0 y = y, \quad L_1 y = r_1((L_0 y)')^\alpha, \quad L_2 y = r_2(L_1 y)', \quad L_3 y = (L_2 y)'$$

on I. Hence (1.1) can be written as

$$L_3 y(t) + \left(\frac{p(t)}{r_1(t)}\right) L_1 y(t) + q(t) f(y(g(t))) = 0.$$

Remark 2.1. If y is a solution of (1.1), then $z = -y$ is a solution of the equation

$$L_3 z(t) + \left(\frac{p(t)}{r_1(t)}\right) L_1 z(t) + q(t) f^*(z(g(t))) = 0,$$

where $f^*(z) = -f(-z)$ and $zf^*(z) > 0$ for $z \neq 0$. Thus, concerning nonoscillatory solutions of (1.1), we can restrict our attention only to solutions which are positive for all large t.

Define the functions

$$R_1(t, t_1) = \int_{t_1}^{t} \frac{\mathrm{d}s}{(r_1(s))^{1/\alpha}}, \quad R_2(t, t_1) = \int_{t_1}^{t} \frac{\mathrm{d}s}{r_2(s)},$$

and

$$R^*(t, t_1) = \int_{t_1}^{t} \left(\frac{R_2(s, t_1)}{r_1(s)}\right)^{1/\alpha} \mathrm{d}s$$

for $t_0 \le t_1 \le t < \infty$. We assume that

$$R_1(t, t_0) \to \infty \quad \text{as} \quad t \to \infty \tag{2.1}$$

and

$$R_2(t, t_0) \to \infty \quad \text{as} \quad t \to \infty. \tag{2.2}$$

In this section, we state and prove the following lemmas which we will use in the proof of our main results.

Lemma 2.2. *Suppose that (1.2) is nonoscillatory. If y is a nonoscillatory solution of (1.1) on $[t_1, \infty)$, $t_1 \ge t_0$, then there exists $t_2 \in [t_1, \infty)$ such that $y(t)L_1 y(t) > 0$ or $y(t)L_1 y(t) < 0$ for $t \ge t_2$.*