Applied Mathematics and Computation 275 (2016) 50–62

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Further results on Cauchy tensors and Hankel tensors

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ARTICLE INFO

MSC:
90C30
15A06

Keywords:
Generalized Cauchy tensor
SOS tensor
Hankel tensor
Positive semi-definiteness
H-eigenvalue

ABSTRACT

In this article, we present various new results on Cauchy tensors and Hankel tensors. We first introduce the concept of generalized Cauchy tensors which extends Cauchy tensors in the current literature, and provide several conditions characterizing positive semi-definiteness of generalized Cauchy tensors with nonzero entries. Furthermore, we prove that all even order generalized Cauchy tensors with positive entries are completely positive tensors, which means every such that generalized Cauchy tensor can be decomposed as the sum of nonnegative rank-1 tensors. We also establish that all the H-eigenvalues of nonnegative Cauchy tensors are nonnegative. Secondly, we present new mathematical properties of Hankel tensors. We prove that an even order Hankel tensor is Vandermonde positive semi-definite if and only if its associated plane tensor is positive semi-definite. We also show that, if the Vandermonde rank of a Hankel tensor $\mathcal{A}$ is less than the dimension of the underlying space, then positive semi-definiteness of $\mathcal{A}$ is equivalent to the fact that $\mathcal{A}$ is a complete Hankel tensor, and so, is further equivalent to the SOS property of $\mathcal{A}$. Thirdly, we introduce a new class of structured tensors called Cauchy–Hankel tensors, which is a special case of Cauchy tensors and Hankel tensors simultaneously. Sufficient and necessary conditions are established for an even order Cauchy-Hankel tensor to be positive definite.

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1. Introduction

Let $\mathbb{R}^n$ be the n dimensional real Euclidean space. Denote the sets of all natural numbers by $\mathbb{N}$. Suppose $m, n \in \mathbb{N}$, $m, n \geq 2$ and denote $[n] = \{1, 2, \ldots, n\}$. It should be noted in advance, we always consider real order m dimension n tensors in this paper.

Tensors (or sometimes called hypermatrices) are the multi-array extensions of matrices. It was recently demonstrated in [13] that most of the problems associated with tensors are, in general, NP-hard. So, it motivates researchers to study tensors with special structure i.e. structured tensors. In the last two or three years, a lot of research papers on structured tensors appeared [3,4,8,9,12,20,28,29,33,37–39,41]. These include M-tensors, circulant tensors, completely positive tensors, Hankel tensors, Hilbert tensors, P-tensors, B-tensors and Cauchy tensors. Many interesting properties and meaningful results of structured tensors have been discovered. For instance, spectral properties of structure tensors, positive definiteness and semi-definiteness of structured tensors were established. Furthermore, some practical applications of structured tensors were studied such as application in

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[1] This author's work was supported by the Natural Science Foundation of China (11171180).
[2] This author's work was partially supported by Australian Research Council.
[3] This author's work was supported by the Hong Kong Research Grant Council (Grant No. PolyU 502111, 501212, 501913 and 15302114).

http://dx.doi.org/10.1016/j.amc.2015.11.051
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stochastic process and data fitting [4,9]. Very recently, authors of [20] studied SOS-Hankel tensors and applied them to the positive semi-definite tensor completion problem.

Among the various structured tensors we mentioned above, there are two particular interesting classes: Cauchy tensors and Hankel tensors. The symmetric Cauchy tensors were defined and analyzed in [3]. In the following discussion, we simply refer it as Cauchy tensors instead of symmetric Cauchy tensors. One of the nice properties of a Cauchy tensor is that its positive semi-definiteness (or positive definiteness) can be easily verified by the sign of the associated generating vectors. In fact, it was proved in [3] that an even order Cauchy tensor is positive semi-definite if and only if each of entries of its generating vector is positive, and an even order Cauchy tensor is positive definite if and only if each entries of its generating vector is positive and mutually distinct.

Hankel tensors arise from signal processing and data fitting [1,9,25]. As far as we know, the definition of Hankel tensor was first introduced in [25]. Recently, some easily verifiable structured tensors related to Hankel tensors were also introduced in [28]. These structured tensors include strong Hankel tensors, complete Hankel tensors and the associated plane tensors that correspond to underlying Hankel tensors. It was proved that if a Hankel tensor is copositive or an even order Hankel tensor is positive semi-definite, then the associated plane tensor is copositive or positive semi-definite respectively [28]. Furthermore, results on positive semi-definiteness of even order strong and complete Hankel tensors were given. However, the relationship between strong Hankel tensors and complete Hankel tensors was not provided in [28]. Later, in [20], it was shown that complete Hankel tensors are strong Hankel tensors; while the converse is, in general, not true.

In this paper, we will provide further new results for Cauchy tensors and Hankel tensors which complements the existing literature. The remainder of this paper is organized as follows. In Section 2, we will first recall some basic notions of tensors are given such as H-eigenvalues, Z-eigenvalues and positive semi-definite tensors. We will also introduce the notion of Vandermonde positive semi-definite tensors, which is a special class of positive semi-definite tensors.

In Section 3, we will first introduce the generalized Cauchy tensors which is an extension of the Cauchy tensors in the literature. Then, we provide complete characterization for positive semi-definite generalized Cauchy tensors with nonzero entries. We also present sufficient and necessary conditions guaranteeing an even order generalized Cauchy tensor with nonzero entries to be completely decomposable. After that, it is proven that all even order generalized Cauchy tensors with positive entries are SOS (sum-of-square) tensors if and only if they are completely positive tensors, which means every generalized Cauchy tensor with positive entries can be written as a sum of nonnegative rank-1 tensors. Furthermore, we prove that the Hadamard product of two positive semi-definite Cauchy tensors are still positive semi-definite tensors. And the nonnegativity for H-eigenvalues of nonnegative Cauchy tensors are testified.

In Section 4, we provide further new properties of Hankel tensors. We prove that the associated plane tensor of an even order Hankel tensor is positive semi-definite if and only if the Hankel tensor is Vandermonde positive semi-definite. Using this conclusion, we give an example to show that, for higher dimensional Hankel tensors, the associated plane tensor is positive semi-definite but the Hankel tensor failed. We also show that, if the Vandermonde rank of a Hankel tensor $\mathcal{A}$ is less than the dimension of the underlying space, then positive semi-definiteness of $\mathcal{A}$ is equivalent to the fact that $\mathcal{A}$ is a complete Hankel tensor, and so, is further equivalent to the SOS property of $\mathcal{A}$.

In Section 5, we introduce Cauchy–Hankel tensors, which are natural extensions of Cauchy–Hankel matrices. The class of Cauchy–Hankel tensors is a subset of Cauchy tensors [3] and Hankel tensors [9,20,28] simultaneously. We provide a checkable sufficient and necessary condition for an even order Cauchy–Hankel tensor to be positive definite. We also show that an even order Cauchy–Hankel tensor is positive semi-definite if and only if the associated homogeneous polynomial is strict monotonically increasing on the nonnegative orthant $\mathbb{R}^n_+$. Some final remarks are provided in Section 6.

Before we end the introduction section, let us make some comments on the symbols that will be used throughout this paper. Vectors are denoted by italic lowercase letters i.e. $x, y, \ldots,$ and matrices are denoted by capital letters $A, B, \ldots$. Suppose $e \in \mathbb{R}^n$ be all one vectors and let e_i denotes the ith unite coordinate vector in $\mathbb{R}^n$. We use bold letters $\mathbf{0} \in \mathbb{R}^n$ to denote zero vector. Tensors are written as calligraphic capitals such as $\mathcal{A}, \mathcal{T}, \ldots$. Let $\mathcal{I}$ denote the real identity tensor. For $x = (x_1, x_2, \ldots, x_n)^T, y = (y_1, y_2, \ldots, y_n)^T \in \mathbb{R}^n$, then $x \ge y$ $(x \le y)$ means $x_i \ge y_i$ $(x_i \le y_i)$ for all $i \in [n]$. $x^{[m]}$ is defined by $(x_1^m, x_2^m, \ldots, x_n^m)^T$.

2. Preliminaries

A real tensor with order m and dimension n is defined by $\mathcal{A} = (a_{i_1 i_2 \ldots i_m})$, $i_j \in [n], j \in [m]$. If the entries $a_{i_1 i_2 \ldots i_m}$ are invariant under any permutation of the subscripts, then tensor $\mathcal{A}$ is called symmetric tensor. Let $x = (x_1, x_2, \ldots, x_n)^T \in \mathbb{R}^n$. The two forms below will be used in the following analysis frequently:

$$\mathcal{A}x^{m-1} = \left(\sum_{i_2, i_3, \ldots, i_m = 1}^{n} a_{i i_2 \ldots i_m} x_{i_2} \ldots x_{i_m} \right)_{i=1}^{n};$$

$$\mathcal{A}x^m = \sum_{i_1, i_2, \ldots, i_m = 1}^{n} a_{i_1 i_2 \ldots i_m} x_{i_1} x_{i_2} \ldots x_{i_m}.$$

Denote $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n | x \ge \mathbf{0}\}$. If $\mathcal{A}x^m \ge 0$ for all $x \in \mathbb{R}^n_+$, then $\mathcal{A}$ is called copositive. An even order m dimension n tensor $\mathcal{A}$ is called positive semi-definite if for any vector $x \in \mathbb{R}^n$, it satisfies $\mathcal{A}x^m \ge 0$. Tensor $\mathcal{A}$ is called positive definite if $\mathcal{A}x^m > 0$ for all nonzero vectors $x \in \mathbb{R}^n$. From the definition, it is easy to see that, for a positive semi-definite tensor, its order m must be an even

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