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SCW method for solving the fractional integro-differential equations with a weakly singular kernel

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ABSTRACT

In this paper, based on the second Chebyshev wavelets (SCW) operational matrix of fractional order integration, a numerical method for solving a class of fractional integro-differential equations with a weakly singular kernel is proposed. By using the operational matrix, the fractional integro-differential equations with weakly singular kernel are transformed into a system of algebraic equations. The upper bound of the error of the second Chebyshev wavelets expansion is investigated. Finally, some numerical examples are shown to illustrate the efficiency and accuracy of the approach.

1. Introduction

In recent years, fractional calculus has attracted many researchers successfully in different disciplines of science and engineering. The advantage of this subject is that fractional derivative and integral are not a local property. Fractional differential equations and fractional integral equations are used to model a lot of practical problems, such as the field of electromagnetic waves, viscoelasticity, dielectric polarization and diffusion equations [1,2]. Fractional integro-differential equations with a weakly singular kernel have many applications in various areas. These equations appear in the radiative equilibrium [3], heat conduction problem [4], elasticity and fracture mechanics [5] and so on.

There have been several numerical methods for the fractional integro-differential equations. For instance, Taylor expansion method [6], homotopy analysis method [7], hybrid collocation method [8], the variational iteration method [9], reproducing kernel method [10], pseudo spectral method [11] and wavelet methods [12–15]. However, only a few methods are proposed to solve the fractional integro-differential equations with a weakly singular kernel [16].

Wavelet analysis plays a prominent role in different areas of mathematics and engineering. Wavelets permit the accurate representation of a variety of functions and operators, and establish a connection with fast numerical algorithms [17]. The wavelets have been used in the solution of all kinds of equations. Using wavelet numerical method has several advantages: (a) The main advantage is that after discreting the coefficient matrix of algebraic equation is sparsity; (b) The wavelet method is computer oriented, thus solving higher order equation becomes a matter of dimension increasing; (c) The solution is a multi-resolution type; (d) The solution is convergence, even the size of increment may be large. Furthermore, SCW have been used to approximate the solution the integral equations of the second kind [18], fractional differential equations [19,20], fractional integro-differential equations [13,14], weakly singular Volterra integral equations [21] and the calculus of variations

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question [22]. The main objective of the present paper is solving the fractional integro-differential equations with a weakly singular kernel base on SCW operational matrix.

The structure of this paper is as follows: In Section 2, SCW are introduced, the convergence of SCW is given. The SCW operational matrix of fractional integration is introduced in Section 3. In Section 4, we summarize the process of solving the fractional integro-differential equations with a weakly singular kernel based on the SCW operational matrix method. The upper bound of the error of the SCW expansion is proposed in Section 5. Several examples are provided to clarify the approach in Section 6. Concluding remarks are given in the last section.

2. The SCW and their properties

The SCW which defined on the interval $[0, 1)$ have the following form [14,19]:

$$\psi_{nm}(t) = \begin{cases} 2^{\frac{k}{2}}\tilde{\mathrm{U}}_{\mathrm{m}}(2^k t - 2n + 1), & \frac{n-1}{2^{k-1}} \leqslant t < \frac{n}{2^{k-1}}, \\ 0, & \text{otherwise}, \end{cases} \tag{1}$$

where $n = 1, \ldots, 2^{k-1}$ and k is any positive integer, and $\tilde{\mathrm{U}}_{\mathrm{m}}(t) = \sqrt{\frac{2}{\pi}}\mathrm{U}_{\mathrm{m}}(t)$, here the coefficient $\sqrt{2/\pi}$ is used for orthonormality; $\mathrm{U}_{\mathrm{m}}(t)$ is the second Chebyshev polynomials of degree m which respect to the weight function $\omega(t) = \sqrt{1-t^2}$. They are defined on the interval $[-1, 1]$ by the recurrence:

$$U_0(t) = 1, \quad U_1(t) = 2t, \quad U_{m+1}(t) = 2tU_m(t) - U_{m-1}(t), \quad m = 1, 2, \ldots$$

The weight function $\tilde{\omega}(t) = \omega(2t-1)$ has to be dilated and translated as $\omega_n(t) = \omega(2^k t - 2n + 1)$.

A function $f(t)$ defined over [0,1], may be expressed in terms of the SCW as

$$f(t) = \sum_{n=0}^{\infty}\sum_{m\in\mathbb{Z}} c_{nm}\psi_{nm}(t),$$

where the coefficients c_{nm} are given by

$$c_{nm} = (f(t), \psi_{nm}(t))_{\omega_n} = \int_0^1 \omega_n(t)\psi_{nm}(t)f(t)\,\mathrm{d}t.$$

We can approximate the function $f(t)$ by the truncated series

$$f(t) \simeq \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1} c_{nm}\psi_{nm}(t) = C^{\mathrm{T}}\Psi(t), \tag{2}$$

where the coefficient vector C and SCW function vector $\Psi(t)$ are given by

$$C = [c_{10}, c_{11}, \ldots, c_{1(M-1)}, c_{20}, \ldots, c_{2(M-1)}, \ldots, c_{2^{k-1}0}, \ldots, c_{2^{k-1}(M-1)}]^{\mathrm{T}}, \tag{3}$$

$$\Psi(t) = [\psi_{10}, \psi_{11}, \ldots, \psi_{1(M-1)}, \psi_{20}, \ldots, \psi_{2(M-1)}, \ldots, \psi_{2^{k-1}0}, \ldots, \psi_{2^{k-1}(M-1)}]^{\mathrm{T}}. \tag{4}$$

Taking the collocation points as following

$$t_i = \frac{2i-1}{2^k M}, \quad i = 1, 2, \ldots, 2^{k-1}M. \tag{5}$$

We define the SCW matrix $\Phi_{m'\times m'}$ as

$$\Phi_{m'\times m'} = \left[\Psi\left(\frac{1}{2m'}\right), \ \Psi\left(\frac{3}{2m'}\right), \ldots, \Psi\left(\frac{2m'-1}{2m'}\right)\right],$$

where $m' = 2^{k-1}M$.

Also, a function $k(t, s) \in L^2_{\omega_n}([0, 1] \times \ [0, 1])$ can be approximated as

$$k(t, s) = \Psi(t)^{\mathrm{T}} K \Psi(s), \tag{6}$$

where K is $2^{k-1}M \times 2^{k-1}M$ matrix with $k_{ij} = (\psi_i(t), (k(t, s), \psi_j(s)))$. We use the wavelet collocation method to determine the coefficients k_{ij}. Taking collocation points as Eq. (5), we obtain the matrix form of Eq. (6)

$$U = \Phi^{\mathrm{T}}_{m'\times m'} K \Phi_{m'\times m'},$$

where $K = [k_{ij}]_{m'\times m'}, U = [k(t_i, s_j)]_{m'\times m'}$.

We investigate the convergence of the SCW expansion in the following lemma.