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Some orders for operators on Hilbert spaces

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ABSTRACT

In this paper we present some interesting properties of the diamond, (left, right) star and sharp orders for operators on Hilbert spaces.

1. Introduction and notations

Throughout the paper $\mathcal{H}$, $\mathcal{K}$ and $\mathcal{F}$ will denote Hilbert spaces. By $\mathcal{B}(\mathcal{H},\mathcal{K})$ we denote the set of all bounded linear operators from $\mathcal{H}$ to $\mathcal{K}$. Set $\mathcal{B}(\mathcal{H})=\mathcal{B}(\mathcal{H},\mathcal{H})$. For an operator $A\in\mathcal{B}(\mathcal{H},\mathcal{K})$, the symbols $\mathcal{N}(A)$, $\mathcal{R}(A)$ and $rank(A)$, respectively, will denote the null space, the range and the dimension of the range of A. The set of all $n\times n$ matrices will be denoted by M_n.

By $x\otimes y^*$ we denote a rank one operator defined by $(x\otimes y^*)(t)=(t,y)x$. Notice that any rank one operator can be represented in this form for some $x,y\in\mathcal{H}$. The span of vectors $\{x,y,\dots w\}$ will be denoted by $\mathcal{L}\{x,y,\dots w\}$. For a closed subspaces X of $\mathcal{H}$ we use the symbol P_X to denote the orthogonal projection onto X.

If $A\in\mathcal{B}(\mathcal{H},\mathcal{K})$ and there exists some $C\in\mathcal{B}(\mathcal{K},\mathcal{H})$ such that $ACA=A$, we say that C is an inner generalized inverse of A and call the operator A relatively regular (regular). It is well known that A is relatively regular if and only if $\mathcal{R}(A)$ is closed in $\mathcal{K}$. By $\mathcal{B}_{reg}(\mathcal{H},\mathcal{K})$ we denote the set of all relatively regular operators from $\mathcal{B}(\mathcal{H},\mathcal{K})$. The regularity of an operator A is equivalent with the existence of the Moore–Penrose inverse of A. Recall that the Moore–Penrose inverse of $A\in\mathcal{B}(\mathcal{H},\mathcal{K})$ (if it exists) is the unique operator $X\in\mathcal{B}(\mathcal{K},\mathcal{H})$ satisfying the following:

$$(1)\ AXA=A\quad(2)\ XAX=X\quad(3)\ (AX)^*=AX\quad(4)\ (XA)^*=XA. \tag{1}$$

It is denoted by $A^\dagger$. It is well-known that $A^\dagger$ exists for a given A if and only if A is a relatively regular operator.

For $A\in\mathcal{B}(\mathcal{H})$, the group inverse of A (if it exists) is the unique operator $A^\#\in\mathcal{B}(\mathcal{H})$ such that

$$AA^\#A=A,\ A^\#AA^\#=A^\#,\ AA^\#=A^\#A. \tag{2}$$

$A\in\mathcal{B}(\mathcal{H})$ has the group inverse if and only if the Drazin index $ind(A)\le 1$. In that case we say that A is group invertible. Some specific characterization of the group invertible elements from $\mathcal{B}(\mathcal{H})$ can be found in [5,12].

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The core inverse on the set of all matrices of index one and core partial order were recently introduced by Baksalary and Trenkler [2]:

Definition 1.1. A matrix $A^{\oplus} \in \mathbb{C}^{n\times n}$ is a core inverse of $A \in \mathbb{C}^{n\times n}$ if $AA^{\oplus} = P_{\mathcal{R}(A)}$ and $\mathcal{R}(A^{\oplus}) \subseteq \mathcal{R}(A)$.

A generalization of this inverse to the algebra of bounded linear operators on a Hilbert space is the following:

Definition 1.2. For $A \in \mathcal{B}(\mathcal{H})$, an operator $A^{\oplus} \in \mathcal{B}(\mathcal{H})$ is a core inverse of A if

$$AA^{\oplus}A = A,\ \mathcal{R}(A^{\oplus}) = \mathcal{R}(A),\ \mathcal{N}(A^{\oplus}) = \mathcal{N}(A^*).$$

The core inverse of A exists if and only if $ind(A) \le 1$. For more details concerning generalized inverses see [3,16].

2. Preliminaries

The next result of Douglas [9] will be frequently used in the paper:

Theorem 2.1. *Let $A \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ and $B \in \mathcal{B}(\mathcal{F}, \mathcal{K})$. The following conditions are equivalent:*

1. $\mathcal{R}(B) \subseteq \mathcal{R}(A)$.
2. *There is a positive number λ such that $BB^* \le \lambda AA^*$.*
3. *There exists $C \in \mathcal{B}(\mathcal{F}, \mathcal{H})$ such that $AC = B$.*

Let $B \in \mathcal{B}_{reg}(\mathcal{H})$ be arbitrary. Without loss of generality, we can suppose that B has the following matrix representation with respect to the orthogonal decomposition $\mathcal{H} = \mathcal{R}(B) \oplus \mathcal{N}(B^*)$:

$$B = \begin{bmatrix} B_1 & B_2 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} R(B) \\ N(B^*) \end{bmatrix} \to \begin{bmatrix} R(B) \\ N(B^*) \end{bmatrix}. \tag{3}$$

Denote by $B' = [B_1 \quad B_2] : [{\mathcal{R}(B) \atop \mathcal{N}(B^*)}] \to \mathcal{R}(B)$. Since $\mathcal{R}(B)$ is closed and $\mathcal{R}(B) = \mathcal{R}(B')$, we have that B' is right invertible. This implies that, $D = B'(B')^* = B_1B_1^* + B_2B_2^* \in \mathcal{B}(\mathcal{R}(B))$ is an invertible operator. Using formula $B^{\dagger} = B^*(BB^{\dagger})^*$ and invertibility of D, we have that

$$B^{\dagger} = \begin{bmatrix} B_1^*D^{-1} & 0 \\ B_2^*D^{-1} & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(B) \\ \mathcal{N}(B^*) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(B) \\ \mathcal{N}(B^*) \end{bmatrix}. \tag{4}$$

If $ind(B) \le 1$, by the elementary observation we can conclude that B_1 is invertible and

$$B^{\#} = \begin{bmatrix} B_1^{-1} & (B_1^{-1})^2B_2 \\ 0 & 0 \end{bmatrix}. \tag{5}$$

Also, when $ind(B) \le 1$ using formula $B^{\oplus} = B^{\#}BB^{\dagger}$, we get that the core inverse of B is given by

$$B^{\oplus} = \begin{bmatrix} B_1^{-1} & 0 \\ 0 & 0 \end{bmatrix}. \tag{6}$$

In the following lemma we will give a representation for the Moore–Penrose inverse of $A \in \mathcal{B}_{reg}(\mathcal{H})$ given by

$$A = \begin{bmatrix} TB_1 & TB_2 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(B) \\ \mathcal{N}(B^*) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(B) \\ \mathcal{N}(B^*) \end{bmatrix}, \tag{7}$$

where $B \in \mathcal{B}_{reg}(\mathcal{H})$ is given by (3) and $T \in \mathcal{B}(\mathcal{R}(B))$.

Lemma 2.1. *Let $B, A \in \mathcal{B}_{reg}(\mathcal{H})$ be given by (3) and (7), respectively for some $T \in \mathcal{B}(\mathcal{R}(B))$. The Moore–Penrose inverse of A has the form*

$$A^{\dagger} = \begin{bmatrix} B_1^*T^*(TDT^*)^{\dagger} & 0 \\ B_2^*T^*(TDT^*)^{\dagger} & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(B) \\ \mathcal{N}(B^*) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(B) \\ \mathcal{N}(B^*) \end{bmatrix}, \tag{8}$$

where $D = B_1B_1^* + B_2B_2^* \in \mathcal{B}(\mathcal{R}(B))$

Proof. Since

$$AA^* = \begin{bmatrix} TDT^* & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(B) \\ \mathcal{N}(B^*) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(B) \\ \mathcal{N}(B^*) \end{bmatrix},$$

using formula $A^{\dagger} = A^*(AA^*)^{\dagger}$, we get that $A\dagger$ is of the form (8). Notice that regularity of operator A implies (more precisely is equivalent) with the regularity of AA^* which is in turn equivalent with the regularity of TDT^*. □