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A finite difference scheme for semilinear space-fractional diffusion equations with time delay

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ABSTRACT

A linearized quasi-compact finite difference scheme is proposed for semilinear space-fractional diffusion equations with a fixed time delay. The nonlinear source term is discretized and linearized by Taylor's expansion to obtain a second-order discretization in time. The space-fractional derivatives are approximated by a weighted shifted Grünwald–Letnikov formula, which is of fourth order approximation under some smoothness assumptions of the exact solution. Under the local Lipschitz conditions, the solvability and convergence of the scheme are proved in the discrete maximum norm by the energy method. Numerical examples verify the theoretical predictions and illustrate the validity of the proposed scheme.

1. Introduction

We aim at constructing a finite difference scheme for the following semilinear time-delay space-fractional diffusion equation

$$u_t(x,t) = K_\alpha \partial_x^\alpha u(x,t) + f(x,t,u,u_s), \quad x \in \mathbb{R}, \quad 0 \le t \le T, \tag{1.1}$$

$$u(x,t) = \psi(x,t), \quad t \in [-s,0], \quad x \in \mathbb{R}, \tag{1.2}$$

where $u = u(x,t)$, $u_s = u(x,t-s)$, $s > 0$ is a constant time delay, K_α is a positive diffusion coefficient, and ∂_x^α is the Riesz fractional derivative with order α, defined by

$$\partial_x^\alpha u(x,t) = -\frac{1}{2\cos(\frac{\alpha\pi}{2})\Gamma(2-\alpha)} \frac{d^2}{dx^2} \int_{-\infty}^{+\infty} |x-\xi|^{1-\alpha} u(\xi,t)d\xi. \tag{1.3}$$

Here, $1 < \alpha < 2$, and $\Gamma(\cdot)$ denotes the gamma function. In one dimension, the Riesz fractional derivative is equivalent to the fractional Laplacian $-(-\Delta)^{\frac{\alpha}{2}}$ under homogeneous Dirichlet boundary conditions [43], i.e.,

$$\partial_x^\alpha u(x,t) = -(-\Delta)^{\frac{\alpha}{2}} u(x,t) := -\mathcal{F}^{-1}(|\omega|^\alpha \mathcal{F}u(\omega,t)),$$

where $\mathcal{F}$ denotes the Fourier transform, i.e. $\mathcal{F}u(\omega,t) = \int_{-\infty}^{+\infty} e^{i\omega x} u(x,t)dx$. Moreover, the Riesz fractional derivative is a virtually linear combination of the left-side and right-side Riemann–Liouville fractional derivatives, i.e.,

$$\partial_x^\alpha u(x,t) = -\frac{1}{2\cos(\frac{\alpha\pi}{2})}[{}_{-\infty}D_x^\alpha u(x,t) + {}_xD_{+\infty}^\alpha u(x,t)], \tag{1.4}$$

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where ${}_{-\infty}D_x^{\alpha}u(x,t)$ represents the left-side Riemann–Liouville fractional derivatives

$$ {}_{-\infty}D_x^{\alpha}u(x,t) = \frac{1}{\Gamma(2-\alpha)}\frac{d^2}{dx^2}\int_{-\infty}^{x}\frac{u(\xi,t)}{(x-\xi)^{\alpha-1}}d\xi, $$

and ${}_{x}D_{+\infty}^{\alpha}u(x,t)$ denotes the right-side Riemann–Liouville fractional derivatives

$$ {}_{x}D_{+\infty}^{\alpha}u(x,t) = \frac{1}{\Gamma(2-\alpha)}\frac{d^2}{dx^2}\int_{x}^{+\infty}\frac{u(\xi,t)}{(\xi-x)^{\alpha-1}}d\xi. $$

We also refer interested readers to [31] for more details.

In recent decades, fractional differential equations (FDEs) have received significant attention due to their widespread applications in many fields of science and engineering which involves anomalous diffusion mechanism, such as underground environmental problem [13], fluid flow in porous materials [4], anomalous transport in biology [14], etc. Some analytical methods, e.g., the Fourier transform method, the Laplace transform method, the Mellin transform method, and the Green function method, can be used to solve some special FDEs, e.g., linear time-fractional differential equations (TFDEs)[31]. For more information of the theory of FDEs, we refer to [8,9]. As most FDEs cannot be solved well by analytical techniques, resorting to numerical methods is an inevitable option. Up to now, there have been many numerical methods for solving FDEs see finite difference methods [6,27,34], finite element methods [17,41], spectral methods [23,38,46], matrix methods [30,32], and also some other numerical techniques [28,29]. Most numerical methods focus on linear FDEs, while studies for nonlinear FDEs are very limited, e.g., [11,22,47,48].

Time delay occurs frequently in realistic world and it has been considered in numerous mathematical models, e.g., automatic control systems with feedback [33], population dynamics [19,21]. Very recently, time delay has been used in modeling HIV infection of $CD4^+T$-cells to describe the time between infection of $CD4^+T$-cells and the emission of viral particles on a cellular level [7,44]. The constant time delay introduces the specific influence of information at $t-s$ on the solution at the given moment t. Note that we need to define an infinite-dimensional set of initial conditions between $t=-s$ and $t=0$. Thus even a single linear delay differential equation is an infinite-dimensional problem [19]. Moreover, the initial history has another impact when we numerically solve a differential equation with delay. Take the equation $y'(t) = -y(t-1), t \geq 0, y(t) = 1, t \in [-1, 0]$, for instance. We have

$$ \begin{cases} y(t) = 1-t, & t \in [0,1) \\ y(t) = 1-t+\frac{1}{2}(t-1)^2, & t \in [1,2) \end{cases} $$

in this particular example, and it is straightforward to check that $y'(0^-) \neq y'(0^+)$ and $y''(1^-) \neq y''(1^+)$, see details in [10]. More generally, the jump in $y'(t)$ at $t=0$ propagates to a jump in $y^{(n+1)}(t)$ at time $t = n\tau,\ n = 1, 2, \ldots,$ which is a feature of delay differential equations. When constructing numerical methods, the discontinuities in low-order derivatives require special attention, since numerical methods for differential equations are intended for problems with solutions that have several continuous derivatives. For theory and numerical methods of delay differential equations, see [2,3]. Due to the increasing application in many fields, such as finance, biology, etc., many numerical methods have also been derived for partial differential equations with time delay [1,15,16,24–26,37].

For the theoretical analysis of fractional differential equations with time delay, we refer to [20]. With respect to numerical methods, a number of works have appeared in the literatures, while most of them focus on time-fractional delay differential equations, e.g., a composite trapezoidal quadrature formula with the predictor–corrector technique [5], the iterative algorithm [40], the Chebyshev spectral method for a fractional order logistic differential equation with two delays [33], etc. Very recently, a framework of spectral and spectral element methods has been derived for time-fractional differential equations with some types of time delay in [45].

In this paper, we propose a high-order linearized finite difference scheme for the semilinear space-fractional time-delay Eqs. (1.1) and (1.2). To determine the boundary conditions, we would confine our study to the problem with analytical solution polynomially decaying to zero when $|x| \to \infty$. In this situation we can truncate the original problem on a bounded computational domain and take the homogeneous Dirichlet conditions.

To obtain time discretizations of second-order accuracy, we make an acceptable assumption that the solution has piecewise smooth derivatives related to variable t in subintervals $(n\tau, (n+1)\tau), n = 0, 1, 2, \ldots,$ instead of the requirement of high-order derivatives on the whole interval $[0, T]$. The similar assumption has also been made in [35]. We approximate the Riesz fractional derivatives in (1.1) by a weighed shifted Grünwald–Letnikov difference formula (WSGD) proposed in [12], which is of fourth-order accuracy. For approximations based on the Grünwald–Letnikov formula and their applications, we refer to [27,36,49]. Nevertheless, in these literatures, the authors mainly pay their attention to linear equations, which motivates us to consider the semilinear Eqs. (1.1) and (1.2). To linearize the nonlinear source term, we use Taylor's expansion on the nonlinear part and derive a linearized finite difference scheme.

In order to obtain the theoretical convergence order in space of the proposed scheme, we assume that the analytical solution of (1.1) and (1.2) is smooth enough related to its spatial variable, and polynomially decays to zero when $|x| \to \infty$. Numerical results in both Examples 4.1 and 4.2, especially in Example 4.2, where the analytical solution is not available to ensure these smoothness assumptions being satisfied, show the efficiency of the proposed scheme for the space fractional diffusion equations with nonlinear source terms and time delay. However, we still need to point out that the smoothness assumptions related to the spatial variable of our proposed scheme is restrictive; see Remark 2.3. In fact, if the assumptions cannot be satisfied, the convergence rate in space of numerical solutions will lose; see numerical examples in [12].