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On the nonlinear self-adjointness of a class of fourth-order evolution equations

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ABSTRACT

In this paper we study the property of nonlinear self-adjointness for a class of nonlinear fourth-order equation. It is shown that if an equation itself is a conservation law then it possesses the property of nonlinear self-adjointness and hence it can be rewritten in an equivalent strictly self-adjoint form. The property of nonlinear self-adjointness is used to obtain all nontrivial conservation laws for the class under consideration.

1. Introduction

In this paper we consider the following fourth order nonlinear evolution equation

$$u_t + (f(u)u_{xxx} + g(u)u_x)_x = 0, \tag{1}$$

with $f \neq 0$ and g smooth functions of u, that is a mathematical model for many physical problems. For example, it appears in the description of the motion of a thin film of viscous liquid (see [3] and the references therein). Moreover, for $f(u) = 1$ it reduces to the well studied Cahn–Hilliard equation [5,11], and describes diffusion for decomposition of a one-dimensional binary solution. Another important equation of this class is, for $f(u) = 1$ and $g(u) = 1 + 2u$, the Childress–Spiegel equation which appears in the study of biofluids [6], solar convection [7] and binary alloys [23].

Using [9,10] and [25] it is possible to find the group classification and similarity solutions of equations of class (1).

Here we study the property of nonlinear self-adjointness [17] for the class (1), in order to obtain nontrivial conservation laws. The conditions for self-adjointness (and its generalizations) for the class of nonlinear differential equations and conservation laws obtained with this procedure can be found in some recent papers (e.g. [4,8,15,16,24,26,27]).

The construction of conservation laws have always been of considerable interest in science. Several ideas and approaches have been developed for constructing conservation laws. One of them is to use the direct method, that is to derive a conservation law for a differential equation by using its definition. Laplace was the first one who used this idea in 1798 for derivation of the well-known Laplace vector of the two-body Kepler problem. In [1,2] the interested reader can find a general treatment of the direct construction method. The modification of the most direct method with usage of equivalence with respect to symmetry or equivalence transformations can be found in [18,21,28]. An elegant algorithm for finding conservation laws is by means of Noether's theorem, but it possesses a strong limitation: it depends upon the knowledge of a suitable Lagrangian. Recently, by using the concept of strict self-adjointness introduced in [13], Ibragimov [14] proved a Noether-type theorem, overcoming the main restrictions of the Noether's approach. The generalization of strict self-adjointness in the concept of nonlinear self-adjointness

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of nonlinear equations [17] was introduced to extend the possibilities of this method. Many other powerful methods have been developed for the construction of conservation laws. In [19] an interesting comparison of different approaches can be found.

The paper is organized as follows. In the next section we study the property of nonlinear self-adjointness for the class (1) and we demonstrate that if an equation itself is a conservation law then it possesses the property of nonlinear self-adjointness with a constant substitution and hence it can be rewritten in the equivalent strictly self-adjoint form. The nontrivial conservation laws for the class under consideration are found in Section 3.

2. On the nonlinear self-adjointness

We start with the essential elements on nonlinear self-adjointness used in this paper. Let

$$F(t, x, u, u_{(1)}, \dots, u_{(s)}) = 0, \tag{2}$$

be a nonlinear partial differential equation of order s, with t, x independent variables, u the dependent variable and $u_{(k)}$ the set of the partial derivatives of kth order of u. In agreement with [13] (see also [14]) the *formal Lagrangian* for Eq. (2) is defined by

$$\mathcal{L} = vF(t, x, u, u_{(1)}, \dots, u_{(s)}), \tag{3}$$

where v is a new dependent variable. Consequently the *adjoint equation* to (2) has the form [13]

$$F^* \equiv \frac{\delta \mathcal{L}}{\delta u} = 0, \tag{4}$$

where $\frac{\delta}{\delta u}$ is the Euler operator.

We recall the definitions of strict and nonlinear self-adjointness [13,14,17].

Definition 1. A nonlinear differential Eq. (2) is said to be strictly self-adjoint if its adjoint equation (4) is satisfied for all solutions u of Eq. (2) upon a substitution

$$v = u.$$

Definition 2. A nonlinear differential Eq. (2) is said to be nonlinearly self-adjoint if its adjoint equation (4) is satisfied for all solutions u of Eq. (2) upon a substitution

$$v = \phi(t, x, u, u_{(1)}, \dots, u_{(n)}), \quad \phi \neq 0, \tag{5}$$

where ϕ is a nonzero function depending on the independent variables, the dependent variable as well as the partial derivatives of the dependent variable until some order n. The number n, that is the highest order of derivatives involved in ϕ, is called the order of the differential substitution (5).

In [17] the author proved also the following connection between the previous two concepts.

Proposition 1. *Eq. (2) is nonlinearly self-adjoint if and only if it becomes strictly self-adjoint upon rewriting it in the equivalent form*

$$\lambda F(t, x, u, u_{(1)}, \dots, u_{(s)}) = 0, \quad \lambda \neq 0, \tag{6}$$

with an appropriate multiplier λ. In particular this multiplier is linked to the substitution (5) by the relation

$$\phi = u\lambda. \tag{7}$$

In this communication, we are interested to study the property of nonlinear self-adjointeness for the class (1). Then after having introduced the *formal Lagrangian* for the class (1)

$$\mathcal{L} = vG \equiv v[u_t + f(u)u_{xxxx} + f'(u)u_x u_{xxx} + g(u)u_{xx} + g'(u)u_x^2], \tag{8}$$

the adjoint equation to (1) has the form

$$G^* \equiv \frac{\delta \mathcal{L}}{\delta u} = 0, \tag{9}$$

where the Euler operator in this case is

$$\frac{\delta}{\delta u} = \frac{\partial}{\partial u} - D_t\left(\frac{\partial}{\partial u_t}\right) - D_x\left(\frac{\partial}{\partial u_x}\right) + D_x^2\left(\frac{\partial}{\partial u_{xx}}\right) - D_x^3\left(\frac{\partial}{\partial u_{xxx}}\right) + D_x^4\left(\frac{\partial}{\partial u_{xxxx}}\right), \tag{10}$$

with D_t and D_x the total differentiation with respect to t and x respectively.

Taking into account the formal Lagrangian (8), the adjoint equation (9) to Eq. (1) reads

$$G^* \equiv v_{xxxx}f + 3u_x v_{xxx}f' + 3u_{xx}v_{xx}f' + 3u_x u_{xx}v_x f'' + 3u_x^2 v_{xx}f'' + u_x^3 v_x f''' + v_{xx}g - v_t = 0. \tag{11}$$