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On the Graovac–Ghorbani index of graphs

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ABSTRACT

The Graovac–Ghorbani index (ABC_{GG}), was introduced by Graovac and Ghorbani, (2010). In this paper we present some upper and lower bounds on the ABC_{GG} index and characterize the extremal graphs.

1. Introduction

Let $G = (V, E)$ be a simple connected graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G)$, where $|V(G)| = n$ and $|E(G)| = m$. Let $d_G(v_i)$ be the degree of vertex v_i for $i = 1, 2, \ldots, n$. If $d_G(v_i) = 1$, then v_i is called a pendant vertex of G and an edge with a pendant vertex on its one end is called a pendant edge. For $v_i, v_j \in V(G)$, the length of the shortest path between the vertices v_i and v_j is their distance, $d_G(v_i, v_j)$. The maximum distance in the graph G is its diameter, d. The atom-bond connectivity (ABC) index of G, proposed by Estrada et al. in [12], and is defined as

$$ABC(G) = \sum_{v_i v_j \in E(G)} \sqrt{\frac{1}{d_G(v_i)} + \frac{1}{d_G(v_j)} - \frac{2}{d_G(v_i)\, d_G(v_j)}}, \tag{1}$$

where $d_G(v_i)$ is the degree of vertex v_i in G. The ABC index has proven to be a valuable predictive index in the study of the heat of formation in alkanes [11,12]. The mathematical properties of this index was reported in [1,3,8,15,16,20,24,26,31]. For details on degree-based topological indices consult [6,19,22,23,27,29,30] and the references cited therein.

Let G be a connected graph and $e = v_i v_j$ be an edge of G. Define two sets $N_i(e|G)$ and $N_j(e|G)$ are as follows:

$$N_i(e|G) = \{v_k \in V(G) | d_G(v_k, v_i) < d_G(v_k, v_j)\},$$
$$N_j(e|G) = \{v_k \in V(G) | d_G(v_k, v_j) < d_G(v_k, v_i)\}.$$

The number of elements of $N_i(e|G)$ and $N_j(e|G)$ are denoted by $n_i = n_i(e|G)$ and $n_j = n_j(e|G)$, respectively. Thus, n_i counts the number of vertices of G lying closer to the vertex v_i than to vertex v_j. The meaning of n_j is analogous. Vertices equidistant from both ends of the edge $v_i v_j$ belong neither to $N_i(e|G)$ nor to $N_j(e|G)$. Note that for any edge e of G, $n_i \geq 1$ and $n_j \geq 1$, because $v_i \in N_i(e|G)$ and $v_j \in N_j(e|G)$. We now define $n_{v_i v_j}$ and $n'_{v_i v_j}$ as follows:

$$n_{v_i v_j} = \max\{n_i, n_j\} \quad \text{and} \quad n'_{v_i v_j} = \min\{n_i, n_j\} \quad \text{for any edge} \quad v_i v_j \in E(G).$$

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Let $n_{\max}$ and $n_{\min}$ be the maximum and minimum of $n_{v_i v_j}$ and $n'_{v_i v_j}$ for any edge $v_i v_j \in E(G)$, respectively, that is,

$$n_{\max} = \max\{n_{v_i v_j} : v_i v_j \in E(G)\} \quad \text{and} \quad n_{\min} = \min\{n'_{v_i v_j} : v_i v_j \in E(G)\}.$$

Therefore we have $1 \le n_{\min} \le n_{\max} \le n-1$. The vertex Szeged index, $Sz(G)$, is defined as [18]:

$$Sz = Sz(G) = \sum_{v_i v_j \in E(G)} n_i n_j. \tag{2}$$

For details of the vertex Szeged index see [4,10,18].

The vertex Padmakar–Ivan Index is another topological index, denoted by $PI_v(G)$ and is defined as [21]

$$PI_v(G) = \sum_{v_i v_j \in E(G)} (n_i + n_j).$$

Recently, several mathematical properties of $PI_v(G)$ index were established [5,25,32].

In [17], Graovac and Ghorbani define a new version of the ABC index as (recently called Graovac–Ghorbani index)

$$ABC_{GG} = ABC_{GG}(G) = \sum_{v_i v_j \in E(G)} \sqrt{\frac{n_i + n_j - 2}{n_i n_j}}.$$

Lower and upper bounds on the ABC_{GG} index of graphs have been given in [9,17,28]. Two indices $ABC(G)$ and $ABC_{GG}(G)$ have been compared in [7,13,14]. In this paper, we present some upper and lower bounds on ABC_{GG} index of graph and characterize the extremal graphs. We denote by K_n, $K_{1,n-1}$ and P_n, the complete graph, star and path on n vertices, respectively, throughout this paper. Let $DS_{p,q}$ $(p \ge q,\ p+q=n)$ be a double star of order n which is constructed by joining the central vertices of two stars $K_{1,p}$ and $K_{1,q}$. For other undefined notations and terminology from graph theory, the readers are referred to [2].

2. Upper bounds on ABC_{GG} index of graphs

In this section we present an upper bound on ABC_{GG} index of graphs. Graovac and Ghorbani [17] gave the following upper bound on ABC_{GG} index of graphs:

$$ABC_{GG}(G) \le PI_v(v) - 2m. \tag{3}$$

Rostami and Haghighat [28] obtained several upper bounds on ABC_{GG} index of graphs. One of the upper bound is as follows:

$$ABC_{GG}(G) < \sqrt{PI_v(G) + m\,(m-3)}. \tag{4}$$

The well-known Lagrange identity is as follows:

Lemma 2.1. *Let $(a) = (a_1, a_2, \ldots, a_n)$ and $(b) = (b_1, b_2, \ldots, b_n)$ be two set of real numbers. Then*

$$\sum_{i=1}^{n} a_i^2 \sum_{j=1}^{n} b_j^2 - \left(\sum_{i=1}^{n} a_i b_i\right)^2 = \sum_{i<j} (a_i b_j - a_j b_i)^2.$$

Lemma 2.2. *For positive integer $n \ge 19$,*

$$\left(\frac{1}{\sqrt{n-1}} - \frac{1}{\sqrt{2\,(n-2)}}\right)^2 \ge \frac{1}{2\,(n-1)\,(n-3)}.$$

Proof. We have to prove that

$$\frac{2(n-3)-1}{2\,(n-1)\,(n-3)} + \frac{1}{2\,(n-2)} \ge \frac{\sqrt{2}}{\sqrt{(n-1)\,(n-2)}},$$

that is,

$$\frac{3\,n^2 - 15\,n + 17}{2\,(n-1)\,(n-2)\,(n-3)} \ge \frac{\sqrt{2}}{\sqrt{(n-1)\,(n-2)}},$$

that is,

$$3\,n^2 - 15\,n + 17 > 2.82\,(n^2 - 4\,n + 3) > 2\,\sqrt{2}\,(n-3)\,\sqrt{(n-1)\,(n-2)},$$

that is,

$$n^2 - 20.8\,n + 47 > 0,$$

which is true for $n \ge 19$. This completes the proof of the lemma. □