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The Pearcey integral in the highly oscillatory region

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ABSTRACT

We consider the Pearcey integral $P(x, y)$ for large values of $|y|$ and bounded values of $|x|$. The integrand of the Pearcey integral oscillates wildly in this region and the asymptotic saddle point analysis is complicated. Then we consider here the modified saddle point method introduced in [Lopez, Pérez and Pagola, 2009] [4]. With this method, the analysis is simpler and it is possible to derive a complete asymptotic expansion of $P(x, y)$ for large $|y|$. The asymptotic analysis requires the study of three different regions for $\arg y$ separately. In the three regions, the expansion is given in terms of inverse powers of $y^{2/3}$ and the coefficients are elementary functions of x. The accuracy of the approximation is illustrated with some numerical experiments.

1. Introduction

The mathematical models of many short wavelength phenomena, specially wave propagation and optical diffraction, contain, as a basic ingredient, oscillatory integrals with several nearly coincident stationary phase or saddle points. The uniform approximation of those integrals can be expressed in terms of certain canonical integrals and their derivatives [2,9]. The importance of these canonical diffraction integrals is stressed in [7] by means of a very appropriate sentence: *The role played by these canonical diffraction integrals in the analysis of caustic wave fields is analogs to that played by complex exponentials in plane wave theory*.

Apart from their mathematical importance in the uniform asymptotic approximation of oscillatory integrals [6], the canonical diffraction integrals have physical applications in the description of surface gravity waves [5,10], bifurcation sets, optics, quantum mechanics and acoustics (see [1, Section 36.14] and references there in).

In [1, Chapter 36] we can find a large amount of information about these integrals. First of all, they are classified according to the number of free independent parameters that describe the type of singularities arising in catastrophe theory, that also corresponds to the number of saddle points of the integral. The simplest integral with only one free parameter, that corresponds to the fold catastrophe, involves two coalescing stationary points: the well-known integral representation of the Airy function. The second one, depending on two free parameters corresponds to the cusp catastrophe and involves three coalescing stationary points. The canonical form of the oscillatory integral describing the cusp diffraction catastrophe is given by the Cusp catastrophe or Pearcey integral [1, p.777, Eq. 36.2.14]:

$$\bar{P}(x, y) := \int_{-\infty}^{\infty} e^{i(t^4+xt^2+yt)}\,dt. \qquad (1)$$

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This integral was first evaluated numerically by using quadrature formulas in [8] in the context of the investigation of the electromagnetic field near a cusp. The third integral of the hierarchy is the Swallowtail integral that depends on three free parameters and involves four coalescing stationary points. Apart from the classification of this family of integrals, in [1, Chapter 36] we can find many properties such as symmetries, illustrative pictures, bifurcation sets, scaling relations, zeros, convergent series expansions, differential equations and leading-order asymptotic approximations among others. But we cannot find many details about asymptotic expansions.

The three first canonical integrals: Airy function, Pearcey integral and Swallowtail integral are the most important ones in applications. The first one is well-known and deeply investigated. In this paper we focus our attention in the second one and will consider the third one in a further research. The integral (1) exists only for $0 \le \arg x \le \pi$ and real y. As it is indicated in [7], after a rotation of the integration path through an angle of $\pi/8$ that removes the rapidly oscillatory term e^{it^4}, the Pearcey integral may be written in the form $\bar{P}(x,y) = 2e^{i\pi/8}P(xe^{-i\pi/4}, ye^{i\pi/8})$, with

$$P(x,y) := \int_0^\infty e^{-t^4-xt^2}\cos(yt)dt. \tag{2}$$

This integral is absolutely convergent for all complex values of x and y and represents the analytic continuation of the Pearcey integral $\bar{P}(x,y)$ to all complex values of x and y [7]. Therefore, it is more convenient to work with the representation (2) of the Pearcey integral.

We can find in the literature several asymptotic expansions of $P(x, y)$ in different regions of (x, y). In [3] we can find an asymptotic expansion of the Pearcey integral when (x, y) are near the caustic $8x^3 - 27y^2 = 0$ that remains valid as $|x| \to \infty$. The expansion is given in terms of Airy functions and its derivatives and the coefficients are computed recursively. We refer the reader to [3] for further details.

An exhaustive asymptotic analysis of this integral can be found in [7]. In particular, a complete asymptotic expansion for large $|x|$ is given in [7] by using asymptotic techniques for integrals applied to the integral (2). The integral $P(x, y)$ is also analyzed in [7] for large $|y|$. But the analysis derived from the standard saddle point method is cumbersome and only the first order term of the asymptotic expansion is given. Then, we are interested here in the derivation of a complete asymptotic expansion of $P(x, y)$ for large values of $|y|$.

In the following section we analyze the saddle point features of the Pearcey integral for large $|y|$. In Section 3 we derive a complete asymptotic expansion of $P(x, y)$ for large $|y|$. Section 4 contains some numerical experiments. Through the paper we use the principal argument $\arg z \in (-\pi, \pi]$ for any complex number z.

2. The saddle point analysis of the Pearcey integral

Because $P(x, y) = P(x, -y)$, without loss of generality, we may restrict ourselves to the half plane $\Re y \ge 0$. We write the Pearcey integral $P(x, y)$ in the form

$$P(x,y) = \frac{1}{2}\int_{-\infty}^{\infty} e^{-u^4-xu^2+iyu}du.$$

Define $\theta := \arg y$ (it is restricted to $|\theta| \le \pi/2$). After the change of variable $u = ty^{1/3} = t|y|^{1/3}e^{i\theta/3}$ we find that

$$P(x,y) = \frac{y^{1/3}}{2}\int_{-\infty e^{-i\theta/3}}^{\infty e^{-i\theta/3}} e^{|y|^{4/3}f(t)-xy^{2/3}t^2}dt, \tag{3}$$

with phase function $f(t) := e^{4i\theta/3}(it - t^4)$. This phase function has three saddle points:

$$t_0 = -\frac{i}{4^{1/3}}, \quad t_1 = \frac{e^{i\pi/6}}{4^{1/3}}, \quad t_2 = \frac{e^{5i\pi/6}}{4^{1/3}}.$$

From the steepest descent method [11, Chapter 2], [4], we know that the asymptotically relevant saddle points are those ones for which the integration path $C := \{re^{-i\theta/3}; -\infty < r < \infty\}$ in (3) can be deformed to a steepest descent path (or union of steepest descent paths) that contains a saddle point (or several saddle points). This is only possible for the saddle points t_1 and t_2[1]. Then, following the idea introduced in [4], we rewrite the phase function $f(t)$ in the form of a Taylor polynomial at the saddle points t_1 and t_2:

$$\begin{aligned} f(t) &= \frac{3e^{i(4\theta+2\pi)/3}}{4^{4/3}} - \frac{3e^{i(4\theta+\pi)/3}}{2^{1/3}}(t-t_1)^2 - 2^{4/3}e^{i(4\theta/3+\pi/6)}(t-t_1)^3 - e^{i4\theta/3}(t-t_1)^4, \\ f(t) &= \frac{3e^{i(4\theta-2\pi)/3}}{4^{4/3}} - \frac{3e^{i(4\theta-\pi)/3}}{2^{1/3}}(t-t_2)^2 + 2^{4/3}e^{i(4\theta/3-\pi/6)}(t-t_2)^3 - e^{i4\theta/3}(t-t_2)^4. \end{aligned} \tag{4}$$

From [4], we know that it is not necessary to compute the steepest descent paths of $f(t)$ at t_1 and t_2, but only the steepest descent paths of the quadratic part of $f(t)$ at those points, that are simpler: they are nothing but straight lines. The steepest descent path

[1] In principle, it is also geometrically possible to deform the original path to the steepest descent path through t_0. But then, the contribution of the paths that connect the original path with the steepest descent path through t_0 is not negligible; whereas the contribution of the paths that connect the original path with the steepest descent path through t_1 and t_2 is negligible, as we show below in formula (5).