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A well balanced and entropy conservative discontinuous Galerkin spectral element method for the shallow water equations

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ABSTRACT

In this work, we design an arbitrary high order accurate nodal discontinuous Galerkin spectral element type method for the one dimensional shallow water equations. The novel method uses a skew-symmetric formulation of the continuous problem. We prove that this discretisation exactly preserves the local mass and momentum. Furthermore, we show that combined with a special numerical interface flux function, the method exactly preserves the entropy, which is also the total energy for the shallow water equations. Finally, we prove that the surface fluxes, the skew-symmetric volume integrals, and the source term are well balanced. Numerical tests are performed to demonstrate the theoretical findings.

1. Introduction

Discontinuous Galerkin formulations for hyperbolic conservation laws typically approximate the conservative form of the equations. This automatically guarantees that the resulting method preserves the conserved quantities discretely. According to the Lax–Wendroff theorem it follows that such a discretisation approximates the right shock speeds, which is of fundamental importance when simulating such phenomena.

When deriving a discontinuous Galerkin method, one needs to choose several ingredients often with many choices available: Two important choices are the specific polynomial basis functions and the numerical quadrature rule used to approximate the inner products of the Galerkin formulation. This discretisation step has a dramatic impact on the efficiency, accuracy and stability of the method. This is especially true when approximating nonlinear hyperbolic problems, where the flux function depends nonlinearly on the conserved quantities. When using a polynomial of degree N in the DG ansatz, it is clear that the flux function no longer belong to the same polynomial space. It could even be that the flux function isn't a polynomial at all, for example if the flux terms are rational with respect to the conserved quantities. Prominent examples where this occurs are the Euler equations of gas dynamics and the shallow water equations. In both cases, the fluxes are rational and hence when using a polynomial ansatz for the conserved variables, are non polynomial. Non polynomial flux functions have a direct consequence on the stability of the discontinuous Galerkin discretisation. If the inner products are not evaluated exactly, aliasing errors are introduced as no precise L_2-projection is performed anymore. In the general approximation, integrals are not evaluated analytically but by choosing a numerical quadrature rule based on Gauss–Legendre-type methods. However, such integration

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rules are all derived from polynomial interpolation and their exactness is only guaranteed when the integrand belongs to the polynomial space. Consequently, it is not possible to avoid aliasing issues in a standard discontinuous Galerkin approximation of hyperbolic problems with non polynomial flux functions. Although it is possible to increase the number of quadrature points so that the integration errors shrink to machine precision, the number of points needed heavily depends on the nonlinearity of the flux function and may vary from problem to problem (e.g. solutions with shocks or smooth solutions only).

In the above sense, there is no rigorous way to prove stability for standard DG discretisations based on the conservative form of hyperbolic problems with non polynomial flux functions. A similar problem exists for other discretisation types such as spectral methods or finite difference methods. In these communities, an interesting strategy to tackle aliasing in nonlinear problems is to resort to a skew-symmetric formulation of the underlying hyperbolic problem, as for instance proposed by Morinishi [1] for the compressible Euler equations. With such formulations, it is possible to derive finite difference methods that are conservative and preserve kinetic energy, e.g. [2,3] and e.g. [4,5]. Another strategy is to derive formulations which are entropy conservative or entropy stable respectively, see e.g. Tadmor [6,7] and e.g. [8–11]. Tadmor showed however, that entropy conservation (stability) is always related to a skew-symmetric formulation of the problem [12]. Thus, these two approaches are closely linked. In fact, we will show in this work that the skew-symmetric formulation of the shallow water equations yields an entropy conservative discretisation.

The skew-symmetric form of the problem is typically achieved by averaging the conservative form of the equation and the non conservative advection form. This is problematic in the sense that discretisations in this form are not obviously conservative for the conserved quantities. However, as mentioned previously, conservation is an important property for the discretisation to have correct shock speeds. Recently, it was shown that when using diagonal norm summation-by-parts operators to discretise the spatial derivatives, skew-symmetric formulations exactly preserve the conservation quantities [13–15]. In particular, the derivations in Fisher et al. [13] show that skew-symmetric operators based on summation-by-parts derivative matrices are consistent and conservative in the Lax–Wendroff sense. It was noted e.g. in [14] that a discontinuous Galerkin spectral element (DGSEM) operator with Gauss–Lobatto points satisfies all properties of a diagonal summation by parts operator and allows for skew-symmetric discretisations that are exactly conservative. This result was used in [15] to derive a kinetic energy conserving DGSEM formulation for the compressible Euler equations using the skew-symmetric form introduced by Morinishi [1]. Unfortunately, kinetic energy for the Euler equations is not an entropy function so kinetic energy conservation is not sufficient to obtain nonlinear stability. Nonlinear entropy stability was recently achieved by Carpenter et al. [16] by constructing a special DGSEM approach, not based on a skew-symmetric form, but using an equivalence of every diagonal norm summation-by-parts operator to a staggered grid type finite volume method.

There are many variants of the DG method applied to the shallow water equations available in the literature, e.g. [17–24] and more general strategies in the context of hyperbolic systems with nonconservative products, e.g. [25,26]. In this work, we combine a DG type discretisation with the idea of skew-symmetry to derive an entropy conserving and well balanced approximation for the shallow water equations.

We start with a special skew-symmetric formulation in Section 2 for the one dimensional shallow water equations, which we use for our DGSEM formulation in Section 3. We prove in Section 4 exact conservation in the Lax–Wendroff sense and show that by choosing an appropriate numerical flux function, the method exactly preserves the total energy, which is an entropy function for the shallow water equations. Furthermore, we will show that this novel skew-symmetric DGSEM formulation allows formal proof of the well balanced property. Section 5 demonstrates and underlines our theoretical findings with numerical results. Our conclusions are presented in the last section.

2. A skew-symmetric form of the shallow water equations

The standard form of the shallow water equations in one dimension for smooth solutions is

$$(C): \quad h_t + (hv)_x = 0 \tag{1}$$

$$(M1): \quad (hv)_t + (h\,v^2 + g\,h^2/2)_x = -g\,h\,b_x \tag{2}$$

where (C) and $(M1)$ indicate the continuity equation and the momentum balance respectively. The quantity $h = h(x, t)$ denotes the water height measured from the bottom topography $b = b(x)$ with the total height given by $H = h + b$. The constant $g = const$ denotes the gravitational acceleration and $v = v(x, t)$ is the velocity.

Instead of writing $(g\,h^2/2)_x$ in flux form, we use the chain rule $g\,h\,h_x$ to get an alternative second form of the momentum equation $(M2)$

$$(M2): \quad (hv)_t + (h\,v^2)_x + g\,h\,(h + b)_x = 0. \tag{3}$$

We will see in the derivations below that this form of the gravity acceleration term is important to achieve a well balanced approximation.

Starting from (C) and $(M2)$, we derive the advection form (A) of the momentum equation $(M2)$ by using the chain rule for the first two terms and subtracting the continuity equation (C) multiplied by the velocity v

$$(A): \quad h\,v_t + (hv)\,v_x + g\,h\,(h + b)_x = 0. \tag{4}$$