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Approximate Osher–Solomon schemes for hyperbolic systems

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ABSTRACT

This paper is concerned with a new kind of Riemann solvers for hyperbolic systems, which can be applied both in the conservative and nonconservative cases. In particular, the proposed schemes constitute a simple version of the classical Osher–Solomon Riemann solver, and extend in some sense the schemes proposed in Dumbser and Toro (2011) [19,20]. The viscosity matrix of the numerical flux is constructed as a linear combination of functional evaluations of the Jacobian of the flux at several quadrature points. Some families of functions have been proposed to this end: Chebyshev polynomials and rational-type functions. Our schemes have been tested with different initial value Riemann problems for ideal gas dynamics, magnetohydrodynamics and multilayer shallow water equations. The numerical tests indicate that the proposed schemes are robust, stable and accurate with a satisfactory time step restriction, and provide an efficient alternative for approximating time-dependent solutions in which the spectral decomposition is computationally expensive.

1. Introduction

The Osher–Solomon scheme [31] is a nonlinear and complete Riemann solver which enjoys a number of attractive features: it is robust, entropy-satisfying, smooth (i.e., differentiable with respect to its arguments) and has a good behavior when computing slowly-moving shocks. However, it requires the computation of a path-dependent integral in phase space, which makes it very complex and computationally expensive. Due to this difficulties, its practical application has been restricted to certain systems, e.g., the compressible Euler equations [37].

In [19,20], Dumbser and Toro proposed a new version of the Osher–Solomon method in which the integrals in phase space are numerically approximated by means of a Gauss–Legendre quadrature formula. This leads to a scheme much simpler than the original one, which is also applicable to general hyperbolic systems. In particular, the viscosity matrix of the numerical flux is defined as a linear combination of the absolute value matrix of the physical flux evaluated at certain quadrature points. The computation of these absolute value matrices requires the knowledge of the complete eigenstructure of the system. Thus, for systems in which the eigenstructure is not known or difficult to compute, the scheme may be computationally expensive.

In this paper we propose an alternative version of the universal Osher–Solomon scheme in [19,20], in which the absolute value matrices are approximated by means of appropriate functional evaluations of the Jacobian of the flux evaluated at the quadrature points. The only information these schemes require is a bound on the maximum speed of propagation. Several families of approximations have been considered. The first one is based on Chebyshev polynomials, which provide optimal uniform approximations to the absolute value function. As an additional feature, the associated Osher–Solomon schemes admit a Jacobian-free

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implementation. On the other hand, it is well-known that the order of approximation to $|x|$ can be greatly improved by using rational functions instead of polynomials. For this reason, two different families of rational approximations have also been considered, based on Newman [29] and Halley [7] functions. These families of functions have also been considered in the recently introduced RVM schemes (see [12]).

It should be noticed that the proposed approximate Osher–Solomon schemes can be defined for general conservative and nonconservative systems. Moreover, they can also be extended to high-order and multidimensions, following the guidelines in [10,13]. In the conservative case, the schemes have been applied to a number of initial value Riemann problems for ideal gas dynamics and magnetohydrodynamics, to observe their behavior with respect to some challenging scenarios in numerical simulations. On the other hand, the two-layer shallow water equations have been considered to test the schemes in the nonconservative framework, as they constitute a representative example including both source and nonconservative coupling terms. Our numerical tests indicate that the proposed schemes are robust, running stable and accurate with a satisfactory time step restriction. Comparisons with the Osher–Solomon and some other well-known schemes in the literature (e.g., Roe and HLL) have also been performed.

The paper is organized as follows. In Section 2 we give some basic concepts needed to review, in Section 3, the classical Osher–Solomon scheme and its extension proposed by Dumbser and Toro in [19,20]. The proposed approximate Osher–Solomon schemes for conservative hyperbolic systems are then introduced in Section 4. The extension to the nonconservative framework is done in Section 5. Several applications to the Euler and ideal magnetohydrodynamics equations are presented in Section 6, while Section 7 is devoted to the two-layer shallow water equations. Some conclusions are drawn in Section 8. Finally, Appendix A contains some details about the implementation of the approximate Osher–Solomon schemes based on Chebyshev approximations, including its Jacobian-free version.

2. Preliminaries

Consider a hyperbolic system of conservation laws

$$\partial_t w + \partial_x F(w) = 0, \tag{1}$$

where $w(x, t)$ takes values on an open convex set $\mathcal{O} \subset \mathbb{R}^N$ and $F : \mathcal{O} \to \mathbb{R}^N$ is a smooth flux function. We are interested in the numerical solution of the Cauchy problem for (1) by means of finite volume methods of the form

$$w_i^{n+1} = w_i^n - \frac{\Delta t}{\Delta x}(F_{i+1/2} - F_{i-1/2}), \tag{2}$$

where w_i^n denotes the approximation to the average of the exact solution at the cell $I_i = [x_{i-1/2}, x_{i+1/2}]$ at time $t^n = n\Delta t$ (the dependence on time will be dropped unless necessary). We assume that the numerical flux is given by

$$F_{i+1/2} = \frac{F(w_i) + F(w_{i+1})}{2} - \frac{1}{2} Q_{i+1/2}(w_{i+1} - w_i), \tag{3}$$

where $Q_{i+1/2}$ denotes the numerical *viscosity matrix*, which determines the numerical diffusion of the scheme.

The condition of hyperbolicity of system (1) states that the Jacobian matrix of the flux at each state $w \in \mathcal{O}$,

$$A(w) = \frac{\partial F}{\partial w}(w),$$

can be diagonalized as

$$A = PDP^{-1},$$

where $D = \text{diag}\,(\lambda_1, \ldots, \lambda_N)$, λ_i being the eigenvalues of A, and the matrix P is composed by the associated right eigenvalues of A. As it is usual, we denote the positive and negative parts of A, respectively, as

$$A^+ = PD^+P^{-1}, \quad A^- = PD^-P^{-1}, \tag{4}$$

where $D^\pm = \text{diag}\,(\lambda_1^\pm, \ldots, \lambda_N^\pm)$, with $\lambda_i^+ = \max(\lambda_i, 0)$ and $\lambda_i^- = \min(\lambda_i, 0)$. It is clear that $A = A^+ + A^-$. On the other hand, the absolute value of A is defined as

$$|A| = A^+ - A^-. \tag{5}$$

It is interesting to note that Roe's method [34] can be written in the form (3) with viscosity matrix given by $Q_{i+1/2} = |A_{i+1/2}|$, where $A_{i+1/2}$ is a Roe matrix for the system. However, the exact computation of $|A_{i+1/2}|$ requires the characteristic decomposition of $A_{i+1/2}$, which in general is computationally expensive. For this reason, several numerical methods have been developed by using approximations to $|A_{i+1/2}|$ as viscosity matrices; see, e.g., [15,17,23,37,38] and the references therein. A general approach to build such kind of approximations by means of polynomial or rational functions has recently been introduced in [9] and [12]. In particular, it has been shown that a number of well-known schemes in the literature can be viewed as particular cases within this general approach: Roe, Lax–Friedrichs, Rusanov, HLL, FORCE, etc.