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Roe-type schemes for shallow water magnetohydrodynamics with hyperbolic divergence cleaning

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ABSTRACT

We discuss Roe-type linearizations for non-conservative shallow water magnetohydrodynamics without and with hyperbolic divergence cleaning.

1. Introduction

Hyperbolic conservation laws are usually equipped with additional conditions. Most important is the existence of a convex entropy—in the case of shallow water the energy—which singles out the physically relevant solution from a large set of possible weak solutions. Sometimes, especially when there is no convex entropy, or the system degenerates into a weakly or resonant hyperbolic system like shallow water magnetohydrodynamics (SMHD), other laws have to be included to find physical solutions. In the first case (convex entropy), the additional law is introduced to accommodate an additional variable, namely the entropy, which depends on the state variables, but is no state variable itself like the energy in shallow water flows. In the latter case, we have additional partial differential equations for the state variables themselves. In the first case, the additional law is a partial differential equation or inequality of evolution type, usually a conservation law itself, in the latter, it is a first order non-evolutionary constraint. These additional constraints are, as Dafermos points out [7,8], involutions for the underlying system of conservation laws. So the resulting system, which includes both the evolution system and the condition, has more equations than unknowns. If the involution is satisfied by the initial state for the evolution equations, it is satisfied by the solution of the evolution system for all times. Thus, in the continuous setting, the constraint is merely a condition for the initial state. The equations of shallow water magnetohydrodynamics (SMHD) are of this latter type, which is a property inherited from full MHD.

To cope with such constraints, there are different approaches in the literature. First, there are approaches, which are designed to model the constraint numerically. Many of them are done on staggered grids [1,2,11]. Some newer approaches also work on collocated grids [12,13,26–28,37,40,41,43] or in the context of Discontinuous-Galerkin schemes [3]. Usually this class of schemes is referred to as *constrained transport* (CT). Applications to SMHD were provided, e.g., by de Sterck [39] and Rossmanith [36]. The basic strategy for most of these CT-methods is to perform a time step for the conservative evolution equations (standard model). In the second step, the magnetic field is recomputed on the basis of a different equation, quite often on the evolution equation for the vector potential of the magnetic field. As a consequence, the evolution of the magnetic field is, at least formally,

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non-conservative. An alternative approach is the one provided by Fey and Torrilhon [40,41]. It is based on averaging fluxes of neighboring Riemann problems, a strategy that is well known to heavily increase numerical viscosity. With the other CT-methods, it shares the drawback that a change in the underlying scheme—e.g., increasing the order of accuracy—also requires a new development of the constraint preserving strategy. Therefore, we concentrate on a different approach.

A second family of schemes is based on a modification of the system of partial differential equations which makes up the constraint part of the evolution system itself. In the context of plasma physics, a popular approach is to transport the involution term, in this case the divergence of the magnetic field, with the flow velocity. This was first suggested for numerical simulations of full MHD by Brackbill and Barnes [4] and put forward by employing Godunovs full symmetrizable form[1] of the MHD-equations [15] by Powell et al. [32,33]. In [12,13,43], this approach is even combined with constrained transport. Another possibility is to apply a kind of a generalized Lagrange multiplier (GLM) approach [31], a method which can appear in several variants: a Hodge-projection scheme, a parabolic treatment of the involution term as was suggested by Marder [25], a hyperbolic system—the involution term is radiated with an artificial speed out of the computational domain [29,30]—or it results in a treatment of the involution in the manner of a telegraph equation [9,22] (Crockett et al. [6] even combine the Marder approach with a Hodge-projection method). In the context of electromagnetic models and plasma physics, these approaches are usually referred to as *divergence cleaning*. Although possible in principle, the combination of the transport of the involution with the hyperbolic GLM approach is, to our knowledge, not yet described in the literature[2].

It is worth noting that, as Brackbill and Barnes [4] point out, the standard system of MHD is deduced from physical principles together with the assumption that the magnetic field is solenoidal. Without this a priori assumption, one would simply end up with the Powell system. As we pointed out in our earlier work [22–24], the derivation of the Maxwell equations themselves without the neglect of magnetic charge and magnetic current, just on the basis of physical symmetries, would lead to the standard Maxwell equations augmented by the terms of the hyperbolic GLM correction. Thus, from the physical point of view, it is desirable to employ MHD (and SMHD) equations which are based on the combination of Powell and hyperbolic GLM.

In this paper, we develop (and test) Roe-type schemes for hyperbolic systems of SMHD equations. For the conservative standard equations, as introduced by Gilman [14], the characteristic analysis was performed by de Sterck [38] and a Roe-type scheme developed by Rossmanith [36]. Based on this preparatory work, we investigate the hyperbolic structure of the extended SMHD systems, i.e., with Powell correction and/or hyperbolic GLM correction, in the following section. In Section 3, we discuss possibilities to get Roe matrices for these systems. This again will be based on the work by Rossmanith [36]. In Section 3, the resulting schemes are tested at hand of two prototypical test cases and compared to the scheme for the standard equations. Furthermore, we also cover the choice for the artificial wave speed and the numerical viscosity of the waves which radiate the divergence errors.

2. The governing equations, combining Powell with GLM-correction

Here, we discuss the shallow water MHD (SMHD) equations with and without Powell and GLM-correction and their eigensystems. The eigensystems are needed in later sections to set up the Roe-type schemes for the different SMHD models.

2.1. Conservative equations

Before we introduce the GLM-correction, we discuss the standard equations and their modification according to Powell. The latter has the desirable property to be fully hyperbolic and not only resonant hyperbolic.

2.1.1. Without divergence cleaning

In [14], Gilman argues that the classical "shallow water" equations of geophysical fluid dynamics should be useful for studying the global dynamics of the solar tachocline and demonstrates the existence of an MHD analogon that would allow taking into account the strong toroidal magnetic field likely to be present there. So he presents a derivation analogous to that for the classical shallow water equations and comes up with the following system of shallow water magnetohydrodynamics (SMHD) equations:

$$h_t + \nabla \cdot [h\boldsymbol{v}] = 0, \tag{1}$$

$$(h\boldsymbol{v})_t + \nabla \cdot \left[h\boldsymbol{v} \circ \boldsymbol{v} - h\boldsymbol{B} \circ \boldsymbol{B} + \frac{gh^2}{2}\boldsymbol{I} \right] = \boldsymbol{0}, \tag{2}$$

$$(h\boldsymbol{B})_t - \nabla \times [\boldsymbol{v} \times (h\boldsymbol{B})] = \boldsymbol{0}, \tag{3}$$

$$\nabla \cdot (h\boldsymbol{B}) = 0, \tag{4}$$

which inherits most of its behavior from the original MHD-system. The main difference is that, due to the averaging over the third space dimension, the magnetic field $\boldsymbol{B}$ is now replaced by $h\boldsymbol{B}$, where h denotes the height of the present fluid layer, and g is

[1] It is interesting to note that this form was first discovered by Godunov [15] as symmetrizable form of MHD and then rediscovered by Powell et al. [32] as Galilean invariant form of MHD. Since it has an entropy [8,15], one would not need any involution for the system.

[2] But it seems to be in practical use, as we can conclude from several oral communications in the last years.