Applied Mathematics and Computation 272 (2016) 420–439

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Finite-volume schemes for Friedrichs systems with involutions

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A R T I C L E I N F O

Article history:
Available online 7 April 2015

Keywords:
Involutionary systems
Friedrichs systems
Finite-volume schemes
Relaxation-type formulation

A B S T R A C T

In applications solutions of systems of hyperbolic balance laws often have to satisfy additional side conditions. We consider initial value problems for the general class of Friedrichs systems where the solutions are constrained by differential conditions given in the form of involutions. These occur in particular in electrodynamics, electro- and magnetohydrodynamics as well as in elastodynamics. Neglecting the involution on the discrete level typically leads to instabilities.

To overcome this problem in electrodynamical applications it has been suggested in Munz et al. (2000) to solve an extended system. Here we suggest an extended formulation to the general class of constrained Friedrichs systems. It is proven for explicit Finite-Volume schemes that the discrete solution of the extended system converges to the weak solution of the original system for vanishing discretization and extension parameter under appropriate scalings. Moreover we show that the involution is weakly satisfied in the limit. The proofs rely on a reformulation of the extension as a relaxation-type approximation and careful use of the convergence theory for finite-volume methods for systems of Friedrichs type. Numerical experiments illustrate our analytical results.

1. Introduction

In this paper, we study linear systems of balance laws, namely $(m \times m)$-systems of Friedrichs [15] type with $m \in \mathbb{N}$. We consider the spatially d-dimensional case with $d \geq 2$, space coordinates $x = (x_1, \ldots, x_d)^T$, and time $t \geq 0$. For $T > 0$, let $G^1, \ldots, G^d, D : \mathbb{R}^d \times [0, T] \to \mathbb{R}^{m\times m}$ and $f : \mathbb{R}^d \times [0, T] \to \mathbb{R}^m$ be given (matrix-valued) functions. We suppose that the matrices $G^1(x, t), \ldots, G^d(x, t)$ are symmetric for all $(x, t) \in \mathbb{R}^d$. Then the initial value problem for the unknown vector-valued function $u : \mathbb{R}^d \times [0, T] \to \mathbb{R}^m$ takes the form

$$\frac{\partial}{\partial t}u(x,t) + \sum_{i=1}^{d} \frac{\partial}{\partial x_i}\left(G^i(x,t)u(x,t)\right) + D(x,t)u(x,t) = f(x,t), \tag{1.1}$$

$$u(x,0) = u_0(x). \tag{1.2}$$

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http://dx.doi.org/10.1016/j.amc.2015.03.050

Here $u_0 : \mathbb{R}^d \to \mathbb{R}^m$ denotes the initial function. Moreover we require the solution u to satisfy a linear differential side condition of the form

$$\sum_{i=1}^{d} M_i \frac{\partial}{\partial x_i}(u(x,t)) = 0, \qquad ((x,t) \in \mathbb{R}^d \times [0,T)). \tag{1.3}$$

Here M_i, $i = 1, \ldots, d$, are constant $(m \times m)$-matrices. Following the notion of Dafermos [7,8] for the side condition (1.3), we restrict ourselves to involutions.

Definition 1.1. The differential constraint (1.3) is called an **involution** for the system (1.1) if and only if any (weak) solution of (1.1)–(1.2) (weakly) satisfies (1.3), whenever the initial data do so.

Involutions appear frequently in applications. We mention the classical Maxwell system to describe electrodynamical processes (cf. [21]). The divergence of the electrical and magnetical field is constrained in this case. The induction equations in the (in)compressible electro- and magnetohydrodynamical equations provide similar examples but with (x, t)-dependence in the flux (Section 5 below). Solutions of the equations of linear elasticity have to satisfy compatibility conditions on the deformation gradient, which result in an involutionary condition (cf. chap. 5 of [7]) Yet another example is the linear piezoelectrical system (see [24]). In Section 5 we present some of these examples in more detail. Let us mention that involutions of course appear also in the more challenging case of nonlinear conservation laws. Again magnetohydrodynamics [6], electrohydrodynamics, nonlinear elasticity systems, but also Einstein's equations of general relativity are prominent examples.

On the analytical level an involutionary side condition is not problematic. The well-posedness for (1.1)–(1.3) is well known from [7]. By definition the involution (1.3) is satisfied. Also standard numerical schemes are known to converge. However, without consideration of (1.3) in the numerical scheme the residuum in the side condition usually grows with increasing time. In coupled processes this is a typical source of instabilities (cf. [25] and cites therein). Therefore a wide range of stabilization methods has been suggested (e.g. [1,3–5,18,26]).

The motivation for this contribution is the work of Munz et al. [26]. They introduced in particular the so-called hyperbolic Generalized Lagrangian Multiplier Finite Volume Method (GLM-FVM) to compute approximate solutions for Maxwell's system of linear electrodynamics. We formulate this approach for the general problem (1.1)–(1.2) with involution (1.3). While the original approach is motivated by a generalization of a Finite-Element type method [1] for a constrained wave equation we consider the approach as the approximation of (1.1)–(1.3) by an extended *relaxation-type system*.

To be precise let a, $\varepsilon > 0$ and u_0, $\psi_0 : \mathbb{R}^d \to \mathbb{R}^m$ be given. Consider the following initial value problem for the unknown function: $w^\varepsilon : \mathbb{R}^d \times [0,T] \to \mathbb{R}^{2m}$, $w^\varepsilon := (u_1^\varepsilon, \ldots, u_m^\varepsilon, \psi_1^\varepsilon, \ldots, \psi_m^\varepsilon)^T$ satisfying

$$\frac{\partial}{\partial t}u^\varepsilon + \sum_{i=1}^{d} \frac{\partial}{\partial x_i}(G^i(x,t)u^\varepsilon) + M_i^T \frac{\partial}{\partial x_i}\psi^\varepsilon + D(x,t)u^\varepsilon = f(x,t), \tag{1.4}$$

$$\frac{\partial}{\partial t}\psi^\varepsilon + \sum_{i=1}^{d} \frac{M_i}{\varepsilon}\frac{\partial}{\partial x_i}u^\varepsilon + a\psi^\varepsilon = 0, \tag{1.5}$$

and

$$u^\varepsilon(x,0) = u_0(x), \qquad \psi^\varepsilon(x,0) = \psi_0(x). \tag{1.6}$$

We will show in Section 2 that the initial value problem for the extended system (1.4)–(1.6) is well-posed. For $\psi_0 \equiv 0$ we have in particular $u^\varepsilon = u$, a.e., where u is the solution of (1.1)–(1.2). In Section 3 we present the Generalized Lagrangian Multiplier Finite Volume Method (GLM-FVM) for the general system (1.1)–(1.3). For mesh parameter $h > 0$ this gives us the mesh function $u_h^\varepsilon : \mathbb{R}^d \times [0,T] \to \mathbb{R}^m$. The method will be analyzed in Section 4. By careful investigation of the convergence theory from Vila and Villedieu [30] and Jovanovic and Rohde [19] we obtain for ε sufficiently small (see Theorem 4.6)

$$\|u_h^\varepsilon - u\|_{L^2(\mathbb{R}^d \times [0,T];\mathbb{R}^m)} = \mathcal{O}(\varepsilon^{-1/4} h^{1/2}). \tag{1.7}$$

Moreover we will show that a weak constraint error behaves like $\mathcal{O}(\varepsilon^{1/4}h^{1/2})$ (Corollary 4.9). The crucial fact is now that by an appropriate choice $\varepsilon = \varepsilon(h)$ one can control the constraint error and ensure the convergence of the method. This expresses the dissipative character of the approximation (1.4)–(1.6). Up to our knowledge convergence statements on error *and* constraint error have not been derived for any of the existing methods to handle involutionary systems [1,3,5,18,26]. Let us point out that the original ansatz from [26] was motivated by the idea to transport divergence errors simply away from the computational domain. Here we consider the full-space case. Our results show therefore that for appropriate choice of parameters the method does not only advect possible involution errors but also damps them.

The assumptions, definitions, general results on Friedrichs systems and some notation are summarized in Section 2, while Section 3 is devoted to the numerical scheme. Section 4 contains the analysis of the scheme and in particular the proofs of the main convergence theorems (Theorem 4.6 and Corollary 4.9). In the last section we present applications and numerical examples to illustrate the relation between error and constraint error for the GLM-FVM. Moreover we show that the GLM-FVM (in contrast to the standard finite-volume method) is efficient in damping the constraint error even for perturbations of (1.1) that violate the constraint. This observation is important for the practical use of the GLM-FVM, e.g., Maxwell's equations with non-conserved charge loading.