Applied Mathematics and Computation 272 (2016) 440–459

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

The structure of well-balanced schemes for Friedrichs systems with linear relaxation

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ARTICLE INFO

Article history:
Available online 13 May 2015

MSC:
65J10
65N06
65N99

Keywords:
Well balanced schemes
Friedrichs systems
Conservative formulation
Finite volume schemes

ABSTRACT

We study the conservative structure of linear Friedrichs systems with linear relaxation in view of the definition of well-balanced schemes. We introduce a particular global change of basis and show that the change-of-basis matrix can be used to develop a systematic treatment of well-balanced schemes in one dimension. This algebra sheds new light on a family of schemes proposed recently by Gosse (2011). The application to the S_n model (a paradigm for the approximation of kinetic equations) for radiation is detailed. The discussion of the singular case is performed, and the 2D extension is shown to be equal to a specific multidimensional scheme proposed in Buet et al. (2012). This work is dedicated to the 2014 celebration of C.D. Munz' scientific accomplishments in the development of numerical methods for various problems in fluid mechanics.

1. Introduction

Our interest in this work is the mathematical structure of Friedrichs systems with linear relaxation, having in mind the definition of well-balanced schemes which is a hot topic nowadays [4,5,16,17,21]. The particular case of Friedrichs systems of large size is challenging for numerical methods, and quite interesting since such large size systems are commonly encountered in the approximation of kinetic equations by moment methods [6,14]. In this work we will consider the S_n model for radiation or neutrons propagation, which is a paradigm for approximations of kinetic equations [10]. Non linear extensions such as well-balanced schemes for shallow water equations or Euler equations are discussed in [2,11,18,23,24].

Our generic model problem is a linear system with relaxation in two dimensions

$$\partial_t U + \partial_x(A(\mathbf{x})U) + \partial_y(B(\mathbf{x})U) = -R(\mathbf{x})U, \quad U(t,\mathbf{x}) \in \mathbb{R}^n, \quad \mathbf{x} = (x,y) \in \mathbb{R}^2, \tag{1}$$

where the unknown is the function $U(t,\mathbf{x})$. The matrices $A(\mathbf{x}), B(\mathbf{x}), R(\mathbf{x}) \in \mathbb{R}^{n\times n}$ may be functions of the space variable, even if they will take as constant in most of this work. In all applications we have in mind the symmetric part of the relaxation matrix on the right hand side is non negative in the sense that $(V, RV) \geq 0$ for all $V \in \mathbb{R}^n$, that is

$$R + R^t \geq 0. \tag{2}$$

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http://dx.doi.org/10.1016/j.amc.2015.04.085
0096-3003/

These matrices are symmetric for a Friedrichs system, that is $A = A^t$, $B = B^t$. The size n can be arbitrarily large. A more specific example which serves as a guideline in this text is the hyperbolic heat equation with $n = 2$

$$\begin{cases} \partial_t p + \partial_x u = 0, \\ \partial_t u + \partial_x p = -\sigma u. \end{cases} \tag{3}$$

Here $U = (p, u)^t$, $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $R = \begin{pmatrix} 0 & 0 \\ 0 & \sigma \end{pmatrix}$. Such linear systems are representative of the linearization of non linear systems with relaxation. Indeed consider the p-system with friction $\sigma \geq 0$

$$\partial_t \tau - \partial_x u = 0, \quad \partial_t u + \partial_x p(\tau) = -\sigma u.$$

Linearization $\tau = \tau_0 + \varepsilon \tau_1 + \dots$ and $u = u_0 + \varepsilon u_1 + \dots$ around an equilibrium $\partial_x \tau_0 = \partial_t \tau_0 = u_0 = 0$ yields

$$\partial_t \tau_1 - \partial_x u_1 = 0, \quad \partial_t u_1 - c_0^2 \partial_x \tau_1 = -\sigma u_1,$$

where $c_0 > 0$ is the speed of sound. This linear system can be rewritten under the form of the Friedrichs system with linear relaxation using the symmetrized variables $U = (c_0 \tau_1, u_1)^t$ and the matrices $A = -c_0 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $R = \sigma \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$.

Remark 1. Adding gravity as in $\partial_t \tau - \partial_x u = 0$ with $\partial_t u + \partial_x p(\tau) = g - \sigma u$ changes the structure of the linearized equations

$$\partial_t \tau_1 - \partial_x u_1 = 0, \quad \partial_t u_1 - \partial_x (c_0(x)^2 \tau_1) = -\sigma u_1$$

where the speed sound is no more constant $c_0 = c_0(x) = \sqrt{-p'(p^{-1}(xg))}$. For simplicity we will not consider this case hereafter, but we mention in our conclusion how the ideas developed in this work can be adapted without major difficulties.

A standard method for the discretization of problems like (1) relies on the splitting method: first, use a finite volume technique for the approximation of the homogeneous equation $\partial_t U + \partial_x (A(\mathbf{x})U) + \partial_y (B(\mathbf{x})U) = 0$, without right hand side; second, add the right hand side solving $\partial_t U = -R(\mathbf{x})U$. Even if efficient for most cases, a splitting method does not respect by construction the stationary states defined by

$$\mathcal{U} = \{ \mathbf{x} \mapsto U(\mathbf{x});\ \partial_x (A(\mathbf{x})U) + \partial_y (B(\mathbf{x})U) = -R(\mathbf{x})U \}.$$

It has been observed in many cases that splitting methods may generate large numerical errors and unphysical oscillations, which therefore must be controlled in a way or another: a recent theoretical contribution on this topic is in [1]. In this direction the so called well-balanced techniques aim at combining finite volume techniques with the knowledge of the right-hand side so as to obtain new schemes which are exact for initial data in $\mathcal{U}$. We refer to [4,5,17,21] for examples. A recent and comprehensive state-of-the-art is to be found in [16]. Moreover an efficient well-balanced scheme is often the starting point of a rigorous asymptotic preserving scheme, a topic that will not be developed in this work but for which we refer the reader to [5,16,21]. All these new methods are quite complex to construct and to analyze, and therefore are difficult to understand. This is why more mathematical analysis is needed to explore the fundamental structures of these techniques.

The basis of the method used hereafter tries to explore such a structure: it is reminiscent in some sense of the seminal work [7], since we begin to modify the equation in a conservative way so that usual finite volume schemes can be directly used to obtain methods which are exact for initial data in $\mathcal{U}$. For this purpose we will use the dual equation

$$\partial_t V + A^t(\mathbf{x}) \partial_x V + B^t(\mathbf{x}) \partial_y V = R^t(\mathbf{x}) V, \quad V(t, \mathbf{x}) \in \mathbb{R}^n. \tag{4}$$

A fundamental property is

$$\partial_t (U, V) + \partial_x (A(\mathbf{x})U, V) + \partial_y (B(\mathbf{x})U, V) = 0 \tag{5}$$

for all U solutions of the primal equation (1) and V solutions of the dual equation (4). The remarkable fact is that identity (5) is in conservation form. It means that if one has enough knowledge of the solutions of the dual equation (4)

$$\mathcal{V} = \{ \mathbf{x} \mapsto V(\mathbf{x});\ A(\mathbf{x})^t \partial_x V + B(\mathbf{x})^t \partial_y V = R^t(\mathbf{x}) V \},$$

then it is possible to replace the non conservative primal equation (1) by the conservative identity (5). That is instead of analyzing the primal set $\mathcal{U}$, we put the emphasis on the dual set $\mathcal{V}$ of stationary states of (4) which are now test functions. This is the basis of this work.

Concentrating of Finite Volume techniques and in view of formula (5), it is possible to conjecture that any well balanced finite volume solver for the primal equation (1) can be recast as a standard finite volume solver for the conservative formulation (5) (more precisely written as Eq. (10) in the core of the paper). In what follows we more modestly discretize directly (5) with usual conservative finite volume solvers, and deduce well-balanced solvers for the non conservative primal formulation (1). We study two families of solvers, which are natural in our context. For the second family called the two-states solver, we show that it corresponds to a well balanced finite volume scheme based on space localization of the source term at the interfaces, see [16]. But in our approach there is absolutely no need of the localization method. Another asset of this method is the possibility to treat general meshes in higher dimensions. Notice that this approach is not restricted to finite volume techniques since the starting point is a modification of the equation.

This work is organized as follows. In Section 2 we detail in one dimension the structure (5) and propose a new conservative formulation of the initial non conservative problem (1). Section 3 is dedicated to the discretization (still in dimension one)