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A conservative, weakly nonlinear semi-implicit finite volume scheme for the compressible Navier–Stokes equations with general equation of state

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The new numerical method introduced in this paper is dedicated to **Claus-Dieter Munz** at the occasion of his 60th birthday and in honor of his scientific contributions to the field of numerical methods for computational fluid dynamics. His work covers the broad range from low Mach number and nearly incompressible flows to highly compressible flows with strong shock waves.

Keywords:
Staggered semi-implicit finite volume method
Large time steps
Mildly nonlinear system
All Mach number flow solver
General equation of state (EOS)
Compressible Euler and Navier–Stokes equations

ABSTRACT

In the present paper a new efficient semi-implicit finite volume method is proposed for the solution of the compressible Euler and Navier–Stokes equations of gas dynamics with general equation of state (EOS). The discrete flow equations lead to a mildly nonlinear system for the pressure, containing a diagonal nonlinearity due to the EOS. The remaining linear part of the system is symmetric and at least positive semi-definite. Mildly nonlinear systems with this particular structure can be very efficiently solved with a nested Newton-type technique.

The new numerical method has to obey only a mild CFL condition, which is based on the fluid velocity and not on the sound speed. This makes the scheme particularly interesting for low Mach number flows, because large time steps are permitted. Moreover, being locally and globally conservative, the new method behaves also very well in the presence of shock waves. The proposed algorithm is first validated against the exact solution of a large set of one-dimensional Riemann problems for inviscid flows with three different EOS: the ideal gas law, the van der Waals EOS and the Redlich–Kwong EOS. In the final part of the paper, the method is extended to the two-dimensional viscous case.

1. Introduction

The use of numerical methods for computational fluid dynamics (CFD) is nowadays widespread in scientific and engineering applications. For example, CFD is the backbone in the design process of modern cars, aircraft, engines or wind turbines, but also in the simulation of geophysical and environmental flows, e.g. for the simulation of atmospheric flows, oceanic currents and tides, storm surges, tsunami waves and for the modelling of water flow in rivers and lakes. The basic governing equations in all these different applications can be derived from first principles by considering the conservation of mass, momentum and energy, leading to the so-called Navier–Stokes equations or one of their simplifications, like the Euler equations or shallow water-type equations. A major difference between the various applications is the *Mach number* $M = \|\mathbf{v}\|/c$, which is the ratio between the

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flow velocity **v** and the sound speed *c*. While typical industrial applications present moderate to high Mach numbers with the formation of shock waves, geophysical flows are usually characterized by low to very low Mach numbers. In general, each of these two flow regimes requires the design of different and specific numerical methods.

At the aid of asymptotic analysis it can be shown [36,37,42] that in the low Mach number limit $M \to 0$ of the compressible Navier−Stokes equations, one retrieves the incompressible Navier–Stokes equations with their typical $\nabla \cdot \mathbf{v} = 0$ constraint on the divergence of the velocity field. In the incompressible limit, the pressure is composed of two parts: a spatially constant thermodynamic background pressure that satisfies the equation of state, and the hydrodynamic pressure fluctuations that are governed by an elliptic pressure Poisson equation. It is very challenging to construct numerical methods that apply to both flow regimes, namely to the compressible and to the incompressible one. While semi-implicit methods are the state-of-the-art for the solution of the incompressible Navier–Stokes equations with and without free surface [7,12,19,20,35,56,58], in the compressible case the family of explicit upwind finite difference and Godunov-type finite volume schemes [29,33,34,39–41,44,48,53,54] is more popular. There have been several important contributions to the extension of staggered semi-implicit pressure-based methods to the compressible regime, see for example [15,43,45]. We also would like to refer to the semi-implicit family of high order discontinuous Galerkin finite element schemes proposed in [22–24].

For some applications the ideal gas law is a sufficiently accurate approximation. In other cases, a more complex equation of state is required to account for real gas effects, such as the van der Waals EOS [57], the Redlich−Kwong EOS [57], the Peng−Robinson EOS [46] or an even more complex EOS like the one of real water, vapor and steam [30,60]. It is very important to note that the sound speed in liquids is of the order of 1000–2000 m/s, while it is of the order of only several hundreds of meters per second in gases. In the wet steam region, where liquid and gas phase coexist, the sound speed drops significantly to 1–10 m/s. Therefore, in the case of compressible multi-phase flows that contain at the same time liquid, vapor and wet steam, the local sound speed and thus the local Mach number changes dramatically, from very low values inside the liquid over moderate values inside the vapour to very high Mach numbers inside the wet steam. For such flows, it would therefore be useful to have one numerical method that is able to solve the governing PDE in all flow regimes accurately and efficiently, from very low Mach numbers up to the high Mach number regime. In [27] a high order accurate explicit Godunov-type finite volume scheme for general equations of state was presented. However, being explicit, that algorithm is not efficient in the case of liquid-dominated flows that contain only few local vapour bubbles, since in this case the time step of the scheme is limited by the large speed of sound in the liquid.

It is therefore the aim of this paper to provide a novel pressure-based semi-implicit method for the compressible Euler equations that is able to deal with general equations of state, which is locally and globally conservative and whose time step is only limited by the flow velocity and not by the sound speed. The new method is designed to apply simultaneously to very low Mach number flows as well as to highly compressible flows with shock waves. In the proposed scheme, the density equation as well as the nonlinear convective terms for momentum and kinetic energy are discretized explicitly, while pressure in the momentum equation and velocity in the energy equation are taken implicitly. This removes the stability condition on the sound speed, and requires only a mild restriction of the time step based on the flow velocity. Then, the discrete momentum equation is inserted into the discrete energy equation, leading to a reduced mildly nonlinear system for the pressure. This nonlinear system has the nonlinearity stemming from the equation of state only on the diagonal, while the remaining linear part of the system is symmetric and at least positive semi-definite. Hence, the pressure can be efficiently obtained with the new family of (nested) Newton-type techniques recently introduced and analyzed by Casulli et al. in [8,9,11,17,18]. The method proposed in this paper has several similarities with the one forwarded by Park and Munz [45], but while the latter was using a linear pressure correction equation and thus was restricted to the ideal gas case, the present approach solves a mildly nonlinear system for the pressure and is sufficiently general to handle complex real gas EOS. Concerning semi-implicit asymptotic-preserving schemes for all-Mach number flows and general equations of state, we also would like to refer to the nice work presented by Cordier et al. in [21], which, however, required the solution of a nonlinear system for pressure and enthalpy, while the present scheme proposed in this paper only requires the solution of a set of scalar systems for the unknown pressure. Furthermore, the method proposed in [21] did not employ the novel nested Newton technique of Casulli et al., which makes the scheme presented in this paper particularly simple, robust and efficient.

The rest of the paper is organized as follows: in Section 2 we present the governing differential equations and their numerical approximation in the one-dimensional inviscid case. A thorough validation of one-dimensional inviscid flows is presented in Section 3, while the extension to multi-dimensional viscous flows is presented in Section 4. Some computational results for the multi-dimensional case are shown in Section 5. Finally, in Section 6 we give some concluding remarks.

2. Governing equations and numerical method

2.1. Governing PDE

The compressible Euler equations of gas dynamics, which represent the principles of conservation of mass, momentum and total energy, in one space dimension read as follows:

$$\frac{\partial}{\partial t}\begin{pmatrix}\rho \\ \rho u \\ \rho E\end{pmatrix} + \frac{\partial}{\partial x}\begin{pmatrix}\rho u \\ \rho u^2 + p \\ u(\rho E + p)\end{pmatrix} = 0, \tag{1}$$