Applied Mathematics and Computation 273 (2016) 1–7

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

The inverse eigenvalue problem of structured matrices from the design of Hopfield neural networks

Lei Zhu[a], Wei-wei Xu[b,*]

[a] College of Engineering, Nanjing Agricultural University, Nanjing 210031, PR China
[b] School of Mathematics and Statistics, Nanjing University of Information Science and Technology, Nanjing 210044, PR China

ARTICLE INFO

Keywords:
Structured matrix
Inverse eigenvalue problem
Matrix norm
Optimal approximation
Frobenius norm

ABSTRACT

By means of the properties of structured matrices from the design of Hopfield neural networks, we establish the necessary and sufficient conditions for the solvability of the inverse eigenvalue problem $AX = X\Lambda$ in structured matrix set $\mathcal{SAR}_{\mathcal{J}}{}^n$. In the case where $AX = X\Lambda$ is solvable in $\mathcal{SAR}_{\mathcal{J}}{}^n$, we derive the generalized representation of the solutions. In addition, in corresponding solution set of the equation, we provide the explicit expression of the nearest matrix to a given matrix in the Frobenius norm.

1. Introduction

Let us first introduce some notations and concepts that will be used in the paper. Let $\mathcal{R}^{n\times m}$ be the set of real $n \times m$ matrices and $\mathcal{SR}^{n\times n}$ stand for the set of all $n \times n$ real symmetric matrices, $\mathcal{ASR}^{n\times n}$ the set of all $n \times n$ real anti-symmetric matrices and $\mathcal{OR}^{n\times n}$ the set of all $n \times n$ orthogonal matrices. We denote the transpose, the conjugate transpose and the Moore–Penrose generalized inverse of a matrix A by A^T, A^H and A^+, respectively, and the identity matrix of order n by I_n. We use $\langle A, B\rangle = tr(B^TA)$ to define the inner product of matrices A and B in $\mathcal{R}^{m\times n}$. Without specification, the matrix norm used in the paper is Frobenius norm defined by $\|A\|_F = \sqrt{tr(A^HA)}$.

Throughout this paper, without special statement we assume that matrix J satisfies $J \in \mathcal{OR}^{n\times n}$ and $J^T = J$. A matrix $A \in \mathcal{R}^{n\times n}$ is said to be in $\mathcal{SAR}_{\mathcal{J}}{}^n$ if $A^T = A$ and $(JA)^T = -JA$. We denote the set of the above $n \times n$ structured matrices by $\mathcal{SAR}_{\mathcal{J}}{}^n$. A matrix $A \in \mathcal{R}^{n\times n}$ is said to be in $\mathcal{SR}_{\mathcal{J}}{}^n$ if $A^T = A$ and $(JA)^T = JA$. The set of this kind of $n \times n$ structured matrices is denoted by $\mathcal{SR}_{\mathcal{J}}{}^n$.

Neural networks initially emerged as an attempt to mimic the biological nervous system with respect to both architecture and information processing strategies. Although these networks were initially intended to perform cognitive tasks that have no precise mathematical descriptions, the networks were subsequently found useful in computational problems and as function approximators. In recent times the term neural network or artificial neural network is used for denoting any massively parallel computing architectures that consist of a large number of simple “neural” processors. A survey of the historical background and development of the subject of neural networks can be found in several publications. We restrict the discussion here to the class of Hopfield networks. We describe the design of Hopfield neural networks as follows.

Each neuron in the Hopfield neural networks is represented by an operational amplifier and the relation between the output v_i and input u_i of the ith amplifier is given by a transfer characteristic s with $v_i = s(u_i)$. The input of each amplifier is connected to ground through a resistor p_i in parallel with a capacitor C_i to simulate the delay of the response of a biological neuron. The

* Corresponding author.
E-mail addresses: zhulei@njau.edu.cn (L. Zhu), wwx19840904@sina.com, wwx19840904@yahoo.com.cn (W.-w. Xu).

http://dx.doi.org/10.1016/j.amc.2015.08.089
0096-3003/

differential equation can be written in the form

$$C_i(du_i/dt) = \Sigma T_{ij} v_j - u_i/R_i + I_i, \tag{1}$$

where $1/R_i = 1/\rho_i + 1/\Sigma_j R_{ij}$. The weight T_{ij} represents a conductance connecting neurons i and j. The output voltage v_i of neuron i is related to its input u_i by the relationship

$$v_i = s(u_i).$$

The performance of the network is governed by the global function which is the energy function $\hat{E}$ given by

$$\hat{E} = -\frac{1}{2}\sum_i \sum_j T_{ij} v_i v_j + \sum_i \frac{1}{R_i}\int_0^{v_i} s_i^{-1}(\xi)d\xi - \sum_i I_i v_i.$$

From the above Equations we derive the following vector (matrix) form:

$$C\frac{dU}{dt} = TV - R^{-1}U + I, \quad E = -\frac{1}{2}V^T TV + r^T S - V^T I, \tag{2}$$

where

$$C = \mathrm{diag}(C_1, C_2, \dots, C_n), \quad R = \mathrm{diag}(R_1, R_2, \dots, R_n),$$

$$S_i = \int_0^{v_i} s_i^{-1}(\xi)d\xi, \quad r_i = 1/R_i.$$

In some cases, C, R, E satisfy $CE = ER$ with $C \in \mathcal{SAR}_{\mathcal{J}}{}^n$. One task is to compensate networks model, uncertainties of the networks C for approximating an objective matrix C_0 s.t.,

$$\|C - C_0\| = \min.$$

As we know, matrix inverse eigenvalue problem is finding an $n \times n$ matrix $A \in \mathcal{L}$ such that $Ax_i = \lambda_i x_i, i = 1, 2, \dots, m,$ where $\mathcal{L}$ is a given set of $n \times n$ matrices with some special structures, $x_1, x_2, \dots, x_m (m \le n)$ are given n-vectors and $\lambda_1, \lambda_2, \dots, \lambda_m$ are given constants. Let $X = [x_1, x_2, \dots, x_m]$, $\Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \dots, \lambda_m)$, then the above relation can be written as $AX = X\Lambda$. The inverse eigenvalue problem of the above models can be described as follows.

Problem I. Given $X = [x_1, x_2, \dots, x_m] \in \mathcal{R}^{n\times m}$ and $\Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \dots, \lambda_m) \in \mathcal{R}^{m\times m}$. Find $A \in \mathcal{SAR}_{\mathcal{J}}{}^n$ such that

$$AX = X\Lambda.$$

Problem II. Let S_1 be the solution set of Problem I. Given $C \in \mathcal{R}^{n\times n}$, find $\hat{A} \in S_1$ such that

$$\|\hat{A} - C\|_F = \min_{A\in S_1} \|A - C\|_F.$$

It is noted that Problem I is the matrix inverse eigenvalue problem. The matrix optimal approximation Problem II, that is, finding the nearest matrix in the solution set of Problem I to a given matrix in Frobenius norm, is proposed in the processes of recovery of linear systems due to revising given dates. A preliminary estimate C of the unknown matrix A can be derived by the experiment observation values and the information of statical distribution.

Many authors studied different inverse eigenvalue problems for different choices of $\mathcal{L}$ by using the structure properties of matrices and the singular value decomposition of the matrix for different structured matrices. For example, see the references [1–4,7–11] for details. We also notice that in [10] Zhou et al. studied the solvability conditions and the general expressions of the solution to the inverse problem based on the structure properties of matrices and the singular value decomposition of the matrix for the structured matrices. In this paper we will use a new different method from the one in [10] to consider the inverse eigenvalue problem of structured matrices from the above design of Hopfield neural networks. In [10] the authors mainly used SVD to derive Lemmas 2 and 3 [10], which can deduce the solutions of the problems. However, in this paper we abstract some special properties based on matrix structures, e.g., Lemmas 2.2, 2.6 and 2.7. This can lead to our Theorems 3.1 and 4.1. Comparing with the results in [10] our results in Theorems 3.1 and 4.1 are a little computable because the results in [10] need singular value decomposition to get matrix ‘U’. Hence, by this motivation we consider the above problems in this paper.

The paper is organized as follows. In Section 2 we discuss the structure properties of the set $\mathcal{SAR}_{\mathcal{J}}{}^n$. In Section 3 we establish the solvability conditions and provide the general solution formula for Problem I. In Section 4 we present the expressions of the solution of Problem II. In Section 5 we give numerical algorithm and example to illustrate our results.

2. Properties of the set $\mathcal{SAR}_{\mathcal{J}}{}^n$

In this section, we introduce the properties of the set $\mathcal{SAR}_{\mathcal{J}}{}^n$.