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# Partition of unity methods for approximation of point water sources in porous media

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ABSTRACT

Several partition of unity methods (PUM) are compared on the problem of steady water flow in an aquifer-well system. In order to improve the approximation of a singular behavior of the pressure near the wells, the standard finite element space is enriched with a cut-off fundamental solution to a Laplace problem with a point source on the whole $\mathbf{R}^2$ space. The optimal order of convergence of PUM in terms of $L^2$ norm of the error is demonstrated. The error of adaptive integration is analysed and a new adaptive strategy is proposed. The influence of the choice of the enriched domain is investigated and its impact on the error is demonstrated numerically.

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## 1. Introduction

Large scale mathematical models of the groundwater flow have to deal with the presence of small scale features like wells and fractures that have a significant impact on the whole solution. The standard finite element method can capture these features using $h$ and/or $p$ adaptivity techniques which is payed of by a larger number of degrees of freedom. One possible alternative is the usage of a suitable partition of unity method (PUM) also known as an extended finite element method (XFEM). The idea is to augment the basis $\{\phi_n\}$ of the discrete finite element space with the functions $u_s\phi_n$, where $u_s$ is an a priori known solution in the vicinity of the small scale feature.

In this work, we use PUM on a steady two-dimensional aquifer model containing hydro-geological wells which cause singularities in the solution. We follow the articles [1,2] due to Gracie and Craig, which are, up to our best knowledge, the first work using the XFEM on the well problems. Our primary aim is to compare different PU methods on a similar model. In particular, we use the XFEM and its corrected version (including ramp function and shift), by Fries e.g. in [3], and the SGFEM introduced by Babuška and Banerjee in [4,5]. We measure the convergence of pressure head in $L^2$ norm over the aquifer domain and we compare the used methods. We also investigate the error of the adaptive integration on the enriched elements and propose a more robust adaptive strategy. In addition, we suggest a better choice of the enriched domain based on a tolerance criterion.

Recently, a work of Ladubec et al. [6] was published, where a time dependent two phase problem of $CO_2$ sequestration is solved. It is focused at the coupling of a Poisson-type equation for pressure, discretized using the XFEM approach, and a saturation equation. A numerical example with a geometry setting analogous to our single aquifer test case is presented there.

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Our implementation[1] is done in C++ language using the Deal II [7], the finite element library not supporting any enrichment techniques at the moment.

The paper is organized as follows. The model and its weak formulation are introduced in Section 2. In Section 3, the discretization using different partition of unity methods is presented in detail. An analysis of the quadrature error and rules for a robust adaptive strategy are derived in Section 4. Section 5 discusses the optimal choice of the enriched domain. Section 6 specifies data of the test problem and discusses numerical results, in particular validation of convergence, behavior of the condition number, a set of experiments validating some of the theoretical results and convergence tests. Finally, conclusions and open questions are summarized in Section 7.

## 2. Model

We consider a steady groundwater flow in a system of aquifers (2D models of horizontal geological layers) separated by aquitards. In contrast to Gracie and Craig [1], we suppose the aquitards to be impermeable but we add a more general volume source term. This allows us later to prescribe a specific source term and a corresponding solution since we are mainly interested in the analysis of the numerical method rather than solving a real-world model in this article.

The aquifers are then connected only by wells which act as sources or sinks in the domain of each aquifer. The pressure in the aquifers is further governed by a Dirichlet boundary condition on the outer boundary of every aquifer.

The model is defined as a complex multi-aquifer system to follow our implementation and to see the differences we made in comparison to Gracie and Craig. Although later, we will deal mostly with one aquifer model.

Let $\Theta^m \subset \mathbf{R}^2$ be the domain of the $m$th aquifer, $m = 1, \ldots, M$. The well $w \in \mathcal{W} = \{1, \ldots, W\}$ is represented by an infinite vertical cylinder $B_w$ with center $\boldsymbol{x}_w$ and radius $\rho_w$. We further denote

$$B_w^m = B_w \cap \Theta^m, \quad \text{and} \quad B^m = \bigcup_{w \in \mathcal{W}} B_w^m,$$

for any aquifer $m$ and a well $w$. The actual computational domain of the aquifer $m$ is $\Omega^m = \Theta^m \setminus B^m$. The boundary $\partial\Omega^m$ of the domain consists of the exterior part $\partial\Theta^m = \Gamma_D^m$ and the interior part $\partial B^m$.

Combining the Darcy law and the continuity equation for incompressible fluid, we get a Poisson equation for the pressure head in the $m$th aquifer:

$$\nabla \cdot (-\mathbf{T}^m \nabla h^m) = f^m \qquad \text{on } \Omega^m \subset \mathbf{R}^2,\ \forall m = 1, \ldots, M, \tag{1}$$

which has to be supplied with boundary conditions

$$h^m|_{\Gamma_D^m} = h_D^m, \tag{2}$$

$$(-\mathbf{T}^m \nabla h^m \cdot \boldsymbol{n})|_{\partial B_w^m} = \sigma_w^m (h^m - H_w^m) \qquad \forall w \in \mathcal{W}, \tag{3}$$

where $\mathbf{T}^m$ [m$^2$s$^{-1}$] denotes the transmissivity tensor, $h^m$ [m] is the pressure head, $f^m$ [ms$^{-1}$] stands for the source density, $\boldsymbol{n}$ is the unit outer normal vector of the interior boundary (i.e. pointing to the centers of wells), $\sigma_w^m$ [ms$^{-1}$] denotes the permeability coefficient between $w$th well and $m$th aquifer, and finally $H_w^m$ is the pressure head in the well $w$ at the level of $m$th aquifer. The total flow from the well $w$ to aquitard $m$,

$$Q_w^m = -\int_{\partial B_w^m} \sigma_w^m (h^m - H_w^m) \, \mathrm{d}\boldsymbol{x}$$

satisfies a simple balance equation on the well

$$Q_w^m = Q_{w,in}^m - Q_{w,out}^m = c_w^{m+1}\left(H_w^{m+1} - H_w^m\right) - c_w^m\left(H_w^m - H_w^{m-1}\right), \quad \forall\, m = 1, \ldots, M \text{ and } \forall\, w \in \mathcal{W}, \tag{4}$$

where $Q_{w,in}^m$ is the flow from the upper aquifer $m+1$, $Q_{w,out}^m$ is the flow to the lower aquifer $m-1$, and $c_w^m$ [m$^2$s$^{-1}$] is the permeability of the well $w$ in the segment below the aquifer $m$. In (4), we assume Darcy flow in the well for the simplicity. The bottom of the well $w$ is impermeable, we set $c_w^1 = 0$, $H_w^0 = H_w^1$ there, and we prescribe given pressure $H_w^{M+1}$ at the top (Fig. 1).

### 2.1. Weak formulation

We define the trial space $V$ and the test space $V_0$:

$$V = (H^1(\Omega^m))^M \times \mathbf{R}^{W(M+1)}, \tag{5}$$

$$V_0 = \left(H_0^1(\Omega^m)\right)^M \times \mathbf{R}^{WM}, \tag{6}$$

where $H^1(\Omega^m)$ is the standard Sobolev space and

$$H_0^1(\Omega^m) = \{\varphi \in H^1(\Omega^m); \varphi|_{\Gamma_D^m} = 0\}.$$

[1] https://github.com/Paulie14/xfem_project .

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