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Acyclic and star coloring of P_4-reducible and P_4-sparse graphs

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ABSTRACT

An acyclic coloring of a graph G is a proper vertex coloring such that G contains no bicolored cycles. The more restricted notion of star coloring of G is an acyclic coloring in which each path of length 3 is not bicolored. In this paper, we mainly study on the acyclic and star coloring of P_4-reducible and P_4-sparse graphs. Moreover, we list polynomial-time algorithms for giving an optimal acyclic or star coloring of a P_4-reducible or P_4-sparse graph.

1. Introduction

All graphs considered in this paper are simple, finite and undirected. We follow the terminology and notation of Bondy and Murty [3]. Except the classical vertex coloring and edge coloring, many kinds of colorings have been studied, such as list coloring, complete coloring and injective coloring. In addition, rainbow connection and rainbow vertex-connection are new types of coloring, as described in a recent survey [21] and other studies [6,14,19,20].

Let G be a simple graph. A *k-vertex coloring*, or simply a *k-coloring* of G is a mapping φ from $V(G)$ to $[k] = \{1, 2, \ldots, k\}$. A vertex coloring is *proper* if no two adjacent vertices are assigned the same color. The chromatic number of a graph G, denoted by $\chi(G)$, is the minimum number of colors required in any proper coloring of G. An *acycle coloring* of a graph G is a proper coloring such that G contains no bicolored cycles. A *star coloring* of a graph G is a acyclic coloring of G in which each path of length 3 is not bicolored. The acyclic and star chromatic numbers of G are defined analogously to the chromatic number and are denoted by $\chi_\alpha(G)$ and $\chi_s(G)$, respectively. Obviously, $\chi_\alpha(G) \leq \chi_s(G)$ for every graph G.

A great deal of graph-theoretical research has been conducted on acyclic and star coloring since they were introduced in the early seventies by Grünbaum [11]. Grünbaum proved an upper bound of nine for the acyclic chromatic number of any planar graph G, with $n \geq 6$ in [11]. His upper bound was improved many times [1,17,22] and at last Borodin [4] proved an upper bound of five colors for planar graphs. Alon et al. in [2] showed that acyclic chromatic number of a graph with maximum degree d is $O(d^{\frac{4}{3}})$ as $d \to \infty$. For the star coloring, the first result is that every subcubic graph is 7-star-choosable [11]. And then, Fertin et al. [10] gave the exact value of the star chromatic number of different families of graphs such as trees, cycles, complete bipartite graphs, outer-planar graphs, and two-dimensional grids, and also studied and gave bounds for the star chromatic number of other families of graphs, such as planar graphs, hypercubes, d-dimensional grids ($d \geq 3$), d-dimensional tori ($d \geq 2$), graphs with bounded tree-width and cubic graphs, and also gave an up-bound for any graph G, $\chi_s(G) \leq \lceil 20d^{\frac{3}{2}} \rceil$, where d is the maximum degree of G.

The acyclic and star coloring problems are both *NP*-hard, and most results concerning their complexity on special classes of graphs are negative. In particular, both problems remain *NP*-hard even when restricted to bipartite graphs in [5,7]. In addition, Albertson and Berman [1] showed that the problem of determining whether the star chromatic number is at most 3 is

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NP-complete even for planar bipartite graphs. In the same paper, the authors also showed that it is *NP*-complete to decide whether the chromatic number of a graph G is equal to the star chromatic number of G, even if G is a planar graph with chromatic number 3. Inapproximability results for both problems are given in [12].

Researchers have obtained a few positive algorithmic results for these problems on graphs for which the acyclic or star chromatic number is bounded by a constant. In particular, Skulrattanakulchai [24] gives a linear-time algorithm for finding an acyclic coloring of a graph with maximum degree 3 that uses four colors or fewer, and Fertin and Raspaud [9] give a linear-time algorithm for finding an acyclic coloring of a graph with maximum degree 5 that uses nine colors or fewer. Lyons [18] gave a polynomial-time algorithm for finding an acyclic and star coloring of cographs.

In this paper, we mainly consider the acyclic and star coloring of P_4-reducible and P_4-sparse graphs. In Section 2, we firstly give some basic definitions and known results, which will be used in the following sections. And then we also give the exact values of the acyclic and star chromatic numbers of a spider graph. Section 3 lists a linear-time algorithm to give an acyclic or star coloring of a P_4-reducible graph, and also gives a polynomial time algorithm to find an acyclic or star coloring of a P_4-spare graph.

2. Preliminaries

In this section, we state some basic concepts and symbols, which will be used in the following paper. For other notation and terminology, we refer to [3] and [23].

Let $G = (V, E)$ be a graph with no isolated vertices. A *path* is a graph $P = (V, E)$ of the form

$$V = \{x_0, x_1, \ldots, x_k\} \quad E = \{x_0x_1, x_1x_2, \ldots, x_{k-1}x_k\},$$

where the x_i are all distinct. A path which has k vertices is denoted by P_k. A graph is called a *cograph* if it has no chordless path on four vertices, also called *P_4-free graph*. A graph G is P_4-reducible if no vertex in G belongs to more than one P_4. The class of P_4-sparse graphs was introduced by Hoàng [13] as the class of graphs for which every set of five vertices induces at most one P_4. The classes of P_4-sparse graphs, cographs and P4-reducible graphs have been studied extensively in recent years and have applications in many areas of applied mathematics, computer science and engineering, mainly because of their good algorithmic and structural properties, see [8,15,16].

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be graphs such that $V_1 \cap V_2 = \phi$. The *disjoint union* of G_1 and G_2 is $G_1 \cup G_2 = (V_1 \cup V_2, E_1 \cup E_2)$. The *join* of G_1 and G_2, denoted by $G_1 + G_2$, is the graph obtained by adding all the possible edges between G_1 and G_2, i.e., $G_1 \vee G_2 = (V_1 \cup V_2, E_1 \cup E_2 \cup \{v_1v_2 | v_1 \in V_1, v_2 \in V_2\}$.

For those two graph operations, there is a known result of the acyclic and star coloring.

Lemma 2.1 [18]**.** *The following hold for any graphs G_1 and G_2.*

(i) $\chi_\alpha(G_1 \cup G_2) = max\{\chi_\alpha(G_1), \chi_\alpha(G_2)\}$.
(ii) $\chi_s(G_1 \cup G_2) = max\{\chi_s(G_1), \chi_s(G_2)\}$.
(iii) $\chi_\alpha(G_1 \vee G_2) = min\{\chi_\alpha(G_1) + |V_2|, \chi_\alpha(G_2) + |V_1|\}$.
(iv) $\chi_s(G_1 \vee G_2) = min\{\chi_s(G_1) + |V_2|, \chi_s(G_2) + |V_2|\}$.

Since cographs are the base of P_4-reducible and P_4-sparse graphs, we will firstly introduce the structure of the cographs as the following definition.

Definition 2.1 [8]**.** A graph $G = (V, E)$ is a cograph if and only if one of the following conditions hold:

(i) $|V| = 1$;
(ii) there exist cographs $G_1, G_2, \ldots, G_k$ such that $G = G_1 \cup G_2 \cup \cdots \cup G_k$;
(iii) there exist cographs $G_1, G_2, \ldots, G_k$ such that $G = G_1 \vee G_2 \vee \cdots \vee G_K$.

Using the above structure of cographs, Lyons [18] gave an polynomial-time algorithm for finding an acyclic and star coloring of cographs. Similarly, we will consider the structure of P_4-reducible and P_4-sparse graphs, and in the next sections give the polynomial-time algorithms for acyclic and star coloring. As end this section, we give the following results for acyclic and star coloring of a spider.

A *spider* is a graph whose vertex set can be partitioned into S, C and R, where $S = \{s_1, s_2, \ldots, s_k\}$ $(k \geq 2)$ is a stable set; $C = \{c_1, c_2, \ldots, c_k\}$ is a complete set; s_i is adjacent to c_j if and only if $i = j$ (a *thin spider*), or s_i is adjacent to c_j if and only if $i \neq j$ (a *thick spider*); R is allowed to be empty and if it is not, then all the vertices in R are adjacent to all the vertices in C and non-adjacent to all the vertices in S. Clearly, the complement of a thin spider is a thick spider, and vice versa. The triple (S, C, R) is called the *spider partition*.

Lemma 2.2 [15]**.** *Let G be a spider with spider partition (S, C, R). If R is empty, then $\chi(G) = |C|$; otherwise, $\chi(G) = |C| + \chi(G[R])$.*

For the acyclic and star coloring of a spider, we get the following lemmas.

Lemma 2.3. *Let G be a spider with spider partition (S, C, R). If R is empty, then $\chi_\alpha(G) = |C|$; otherwise, $\chi_\alpha(G) = \chi_\alpha(G[R]) + |C|$.*

Proof. Let G be a spider with spider partition (S, C, R). We will prove the lemma by the following cases.