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A note on some recent fixed point results for cyclic contractions in *b*-metric spaces and an application to integral equations

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ABSTRACT

In this paper we obtain some equivalences between cyclic contractions and non-cyclic contractions in the framework of *b*-metric spaces. Our results improve and complement several recent fixed point results for cyclic contractions in *b* -metric spaces established by George et al. (2015) and Nashine et al. (2014). Moreover, all the results are with much shorter proofs. In addition, an application to integral equations is given to illustrate the usability of the obtained results.

1. Introduction and preliminaries

Fixed point theory is one of the traditional theory in mathematics and has a large number of applications in many branches of nonlinear analysis. It is well known that the celebrated Banach contraction principle [4] is a basic result in fixed point theory, which has been extended in many different directions. One of the most interesting generalizations was given by Kirk et al. [16] in 2003 by introducing the following notion of cyclic representation.

Definition 1.1 [16]**.** Let A and B be nonempty subsets of a metric space (X, d) and $T\colon A\cup B \to A\cup B$. Then T is called a cyclic map if $T(A) \subseteq B$ and $T(B) \subseteq A$.

The following interesting theorem for a cyclic map was given in [16].

Theorem 1.2. *Let A and B be nonempty closed subsets of a complete metric space (X, d). Suppose that $T\colon A\cup B \to A\cup B$ is a cyclic map such that*

$$d(Tx, Ty) \le \lambda d(x, y)$$

for all $x \in A$ and $y \in B$, where $k \in [0, 1)$ is a constant. Then T has a unique fixed point u and $u \in A\cap B$.

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It should be noticed that cyclic contractions (unlike Banach-type contractions) need not be continuous, which is an important gain of this approach. Following the work of Kirk et al., several authors proved many fixed point results for cyclic mappings, satisfying various (nonlinear) contractive conditions. For some results and observations, the reader refers to [20] and [21].

Berinde initiated the concept of almost contractions and obtained several interesting fixed point theorems (see [5]). Here we recall them, but in the context as in [22].

Definition 1.3. Let f and g be two self-mappings on a metric space (X, d). They are said to satisfy almost generalized contractive condition, if there exist a constant $\delta \in (0, 1)$ and some $L \geq 0$ such that for all $x, y \in X$,

$$d(fx, gy) \leq \delta M(x, y) + LN(x, y),$$

where

$$M(x, y) = \max\left\{d(x, y), d(x, fx), d(y, gy), \frac{d(x, gy) + d(y, fx)}{2}\right\}$$

$$N(x, y) = \min\{d(x, fx), d(y, gy), d(x, gy), d(y, fx)\}.$$

Khan et al. [14] introduced the concept of altering distance function as follows.

Definition 1.4. A function φ: $[0, \infty) \to [0, \infty)$ is called an altering distance function if the following properties hold:

(1) φ is continuous and nondecreasing;
(2) $\varphi(t) = 0$ if and only if $t = 0$.

Also, there are some generalizations of usual metric spaces. One well-known generalization is b-metric space (see [3,6]) or metric type space (for short, MTS) called by some authors (see [10,12,13]).

The following definition is introduced in [3] and [6].

Definition 1.5. Let X be a (nonempty) set and $s \geq 1$ be a given real number. A function d: $X \times X \to [0, \infty)$ is called a b-metric on X if, for all $x, y, z \in X$, the following conditions hold:

(b1) $d(x, y) = 0$ if and only if $x = y$;
(b2) $d(x, y) = d(y, x)$;
(b3) $d(x, z) \leq s[d(x, y) + d(y, z)]$ (b-triangular inequality).

In this case, the pair (X, d) is called a b-metric space (metric type space).

Otherwise, for more definitions such as b-convergence, b-completeness, b-Cauchy sequence in b-metric spaces, we see [1–3,6–13,15,17–19,23].

Note that every metric space is a b-metric space (metric type space), but the converse is not necessarily true (see [1,2,6–13,15,17,19,22,23]).

In the sequel we use the following lemma to show the fixed point results in the framework of b-metric spaces.

Lemma 1.6 ([12], Lemma 3.1)**.** Let $\{y_n\}$ be a sequence in a b-metric space (X, d) with $s \geq 1$ such that

$$d(y_n, y_{n+1}) \leq \lambda d(y_{n-1}, y_n)$$

for some $\lambda \in [0, \frac{1}{s})$, and each $n = 1, 2, \ldots$. Then $\{y_n\}$ is a b-Cauchy sequence in (X, d).

Besides that Definition 1.1 authors still introduced the following more general definition:

Definition 1.7 [16]**.** Let X be a nonempty set. Let p be a positive integer, $A_1, A_2, \ldots, A_p$ be nonempty subsets of X, $Y = \cup_{i=1}^{p} A_i$ and T: $Y \to Y$. Then $Y = \cup_{i=1}^{p} A_i$ is said to be a cyclic representation of Y with respect to T if

(i) $A_i (i = 1, 2, \ldots, p)$ are nonempty closed sets;
(ii) $T(A_1) \subseteq A_2, \ldots, T\left(A_{p-1}\right) \subseteq A_p, T(A_p) \subseteq A_1$.

George et al. [7] proved the following fixed point results for cyclic contractions in the setting of b-metric spaces.

Theorem 1.8. Let $\{A_i\}_{i=1}^{p}$ where p is a positive integer, be nonempty closed subsets of a b-complete b-metric space (X, d) with coefficient $s \geq 1$ and suppose that $T : \cup_{i=1}^{p} A_i \to \cup_{i=1}^{p} A_i$ is a cyclical operator that satisfies the condition

$$T(A_i) \subseteq A_{i+1} \textit{ for all } i \in \{1, 2, \ldots, p\}$$

such that

$$d(Tx, Ty) \leq \lambda d(x, y)$$

for all $x \in A_i, y \in A_{i+1}$ and $\lambda \in (0, \frac{1}{s})$.

Then T has a unique fixed point.