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An alternative approach to study nonlinear inviscid flow over arbitrary bottom topography

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ABSTRACT

This paper deals with a new approach to study the nonlinear inviscid flow over arbitrary bottom topography. The problem is formulated as a nonlinear boundary value problem which is reduced to a Dirichlet problem using certain transformations. The Dirichlet problem is solved by applying Plemelj–Sokhotski formulae and it is noticed that the solution of the Dirichlet problem depends on the solution of a coupled Fredholm integral equation of the second kind. These integral equations are solved numerically by using a modified method. The free-surface profile which is unknown at the outset is determined. Different kinds of bottom topographies are considered here to study the influence of bottom topography on the free-surface profile. The effects of the Froude number and the arbitrary bottom topography on the free-surface profile are demonstrated in graphical forms for the subcritical flow. Further, the nonlinear results are validated with the results available in the literature and compared with the results obtained by using linear theory.

1. Introduction

In recent times, the problems involving free-surface fluid flow over submerged obstacles have created various challenges to model various situations occurring in oceanography and atmospheric sciences. Hence, the free-surface flow problems are studied by a large group of researchers for finding the solutions. Earlier the problems of free-surface fluid flow were solved by using linear theories. Kelvin [1] presented a linear theory for the steady form of the fluid flow over submerged obstacle, considering fluid to be inviscid and incompressible. Lamb [2] studied the problem involving flow in a channel and calculated the drag on a cylindrical obstruction. Long [3] linearized the governing equations around uniform upstream flow and obtained finite amplitude wave solutions. Havelock [4] considered the problem in which a dipole moves with a constant velocity underneath the surface of an infinitely deep fluid. He presented a linearized solution which describes the flow about a semicircular obstruction on the bottom of a horizontal canal. A good review on the flow problems is provided in [5–7] and the references therein.

The asymptotic theories involving flow of a fluid in a channel are also popular. Grimshaw and Smyth [8] presented a theoretical study for unsteady flow of a stratified fluid over bottom topography for the case when the flow is near resonance. They pointed out that the flow can be described by a forced Korteweg-de Vries (KdV) equation. This equation is discussed both analytically and numerically for the supercritical flow as well as subcritical flow. Shen et al. [9] obtained the solution of the problem of near critical flow of an ideal fluid over a semicircular as well as semielliptical bump at the bottom using a similar forced KdV

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equation. They have shown that their solution has two branches and there exists a cut-off value of the Froude number below which no solution is possible. They also obtained a branch of hydraulic fall solutions which decrease monotonically from upstream to downstream. Milewski and Vanden-Broeck [10] investigated the time dependent two-dimensional free surface flow past a submerged moving obstacle in a shallow water channel including the effect of surface tension. They solved the problem by using weak nonlinear theory and showed that many unsteady phenomena occur when Froude number and Bond numbers are varying. Chakrabarti and Martha [11] have considered the flow problem in an infinite channel with arbitrary bottom topography and obtained the solution using linear as well as weak nonlinear theory.

Later on, many researchers turned their attention to solve the nonlinear free-surface problems by assuming the free-surface condition in its exact nonlinear form. Solutions of these nonlinear problems are somehow difficult to obtain as compared to the linear ones, especially through the analytical approach. Therefore, many researchers have used various numerical techniques to solve the nonlinear flow problems. For instance, Haussling and Coleman [12] solved the problem involving time-dependent potential flow with the help of boundary fitting technique, in which a curvilinear co-ordinate system is solved numerically. A good review article by Yeung [13] is devoted to the numerical solution of the problem of free-surface fluid flow. Recently, a number of investigators have applied various mathematical techniques to solve nonlinear flow problems. For example, Forbes [14,15] investigated the problem involving drag-free flow over a semi-elliptic obstacle on the stream bed with the help of conformal mapping which is used to satisfy the bottom condition exactly. Forbes and Schwartz [16] considered the flow over a semicircular obstruction in a single layer of fluid and calculated the wave resistance of the semicircle by using the solution at the free surface. Later, Forbes [17] carried out the numerical solution for the free surface flow over a semicircular obstruction attached to the bottom of a running stream. Dias and Vanden-Broeck [18] studied the problem involving free surface flow past a submerged triangular obstacle at the bottom of a channel and solved the problem numerically by series truncation technique. They have determined local solutions to describe the flow near the apex when the height of the triangular obstacle is infinite. Vanden-Broeck [19] analyzed the problem of Forbes and Schwartz [16] numerically by using an integro-differential equation formulation and has shown that the supercritical solution exists only for different values of the Froude number greater than some particular value. He has also shown that the supercritical solution approaches a solitary wave as the size of the obstacle approaches zero. Dias and Vanden-Broeck [20] solved the nonlinear free surface flow past a semicircular obstruction using weakly nonlinear analysis. They restricted their study to the steady flow and they demonstrated that there are supercritical flows with waves downstream. Based on Lie group method, Abd-el-Malek and Amin [21] studied the nonlinear flow of an inviscid fluid over a horizontal bottom. From the above literature, it is found that, in most cases, the specific geometry of the bottom profile has been used to simplify the equations. Thus, the arbitrary bottom profiles are not allowed. Therefore, the fully nonlinear flow over an arbitrary bottom topography remained unsolved. To the best of the authors' knowledge, few articles related to this study are available in the scientific literature. In this context, King and Bloor [22] first studied the fully nonlinear flow over an arbitrary bottom topography. They have used inverse formulation based on a complicated Schwarz–Christoffel transformation [23] to study the flow problem. An alternate method was presented by Belward and Forbes [24] who used a boundary-integral method in the original variables to compute downstream interfacial waves. It is observed from above literature that the nonlinear boundary conditions make the problem more difficult to solve. Hence, it is an endeavour to develop a straightforward method to solve such nonlinear boundary value problem (BVP) for finding its solution.

In this paper, an effort has been made to solve the nonlinear flow over an arbitrary bottom topography using a new approach which is different, simpler and easily apprehended than the methods available in the above literature. The present problem is formulated mathematically in terms of a nonlinear BVP which is reduced to a Dirichlet problem after using certain transformations. After utilizing a suitable application of Plemelj–Sokhotski formulae, the Dirichlet problem is solved. It is found that the solution of the Dirichlet problem depends on the solution of a coupled Fredholm integral equations of the second kind. The coupled integral equations are solved numerically with the help of a modified numerical method to produce free-surface profile. A number of observations are made to demonstrate the effect of nonlinearity. The effects of the Froude number and the arbitrary bottom topography on free-surface profile are discussed for the subcritical flow. The present results are also validated with the results available in the literature and compared with the results in the case of linear ones.

2. Description and formulation of the problem

Here, a two-dimensional potential flow of an inviscid and incompressible fluid is considered. The fluid which is subjected to the downward acceleration of gravity g is flowing from left to right over an arbitrary bottom topography in a channel. At the far upstream, the flow is uniform with speed c and fixed depth H; and at the far downstream, the flow is also uniform with the speed c^* and the depth H^*. Assuming that the flow under study is steady, with everything moving with a constant velocity v, we formulate the present problem as described below. A right-handed rectangular Cartesian co-ordinate system is adopted in which the x-axis is measured along the unperturbed bottom and the y-axis is measured vertically upward. The bottom is denoted by $y = B(x, t)$ and the free surface is denoted by $y = \eta(x, t)$ which is unknown at the outset. The flow domain is shown schematically in Fig. 1.

With the above assumptions, the *equation of continuity* yields the Laplace equation

$$\frac{\partial^2 \Phi^*}{\partial x^2} + \frac{\partial^2 \Phi^*}{\partial y^2} = 0, \quad \text{in the fluid region,} \tag{1}$$

where $\Phi^*(x, y, t)$ represents the velocity potential of the flow.