Applied Mathematics and Computation 273 (2016) 190–200

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

On quasi-periodic properties of fractional sums and fractional differences of periodic functions

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ARTICLE INFO

MSC:
39A23

Keywords:
Discrete fractional calculus
Fractional difference equation
Periodic solution
Asymptotically periodic property
Discrete R transform

ABSTRACT

This article is devoted to the study of discrete fractional calculus; our goal is to investigate quasi-periodic properties of fractional order sums and differences of periodic functions. Using Riemann–Liouville and Caputo type definitions, we study concepts close to the well known idea of periodic function, such as asymptotically periodicity or S-asymptotically periodicity. We use basic tools of discrete fractional calculus. Boundedness of sums and differences of fractional order of a given periodic function is also investigated.

1. Introduction

The ideas of fractional calculus, that is, integrals and derivatives of non integer order, have a long tradition. The birth of this theory comes back to Leibniz's note in his letter to L'Hôpital dated 30 September 1695.

The key which motivated the most used definition of fractional order integral of a given function is the Cauchy formula for n-multiple integrals. Then, using this definition, the fractional order derivative of a function is introduced. These two concepts are the starting point of the theory of differential equations of non integer order, which has received increasing attention during recent years since fractional order differential operators provide an excellent instrument to study memory processes and properties of some materials. For a detailed exposition of fractional calculus, we refer the reader to references [13,16,17].

However, the theory of fractional difference calculus, which involves sums and differences of fractional order, is much less developed. In [14], Miller and Ross initiated the study of discrete fractional calculus. Some notions of fractional order differences can be found in [1], while essentials of fractional difference equations are the subject matter of some recent papers [4–8].

Periodic functions play a major role in mathematics. Indeed, the study of existence of periodic solutions and oscillatory behaviors is one of the most interesting and important topics in qualitative theory of differential equations. Of course, this is due to its implications in pure and applied analysis.

Nevertheless, the definition of periodic function is extremely demanding and then, conditions to guarantee the existence of periodic solutions are sometimes very harsh. For this reason, in the past decades, many authors (see [10,11,15] and references therein) have proposed and studied extensions of the well known idea of periodic function which have shown to be really interesting and useful in some situations.

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http://dx.doi.org/10.1016/j.amc.2015.09.082
0096-3003/

It is an obvious fact that the classical derivative, if it exists, of a periodic function is also a periodic function with the same period. Moreover, the primitive of a periodic function may be periodic. The same holds also for discrete difference and sum operators. Nevertheless, when we consider derivatives or integrals of non integer order, this is not true (see, for example, [2,18]). In [3] , we have studied some properties of fractional integrals and derivatives of periodic functions.

Motivated by references [3,8], we present here the analogous results in the field of discrete fractional calculus. In the next section we introduce some definitions and notations. In Section 3 we prove that the fractional sum or difference of a periodic function is not a periodic function. Moreover, we consider S-asymptotically periodicity properties of discrete fractional operators. In Section 4 asymptotically periodicity properties are studied. Section 5 deals with boundedness of fractional sums and differences and finally, in Section 6 we point out one important difference between continuum and discrete fractional operators.

2. Some basic definitions and results

Throughout the paper, we assume the property of empty sum; that is, if $a > b$ then

$$\sum_{t=a}^{b} f(t) = 0. \tag{1}$$

Given $t \in \mathbb{R}$, we set $\mathbb{N}_t = \{t, t+1, t+2, \dots\}$.

Let us recall the definitions of difference and sum of fractional order, which have been previously introduced in [14].

Definition 2.1. Let $\alpha > 0$, the fractional sum of f of order α with base point $a \in \mathbb{Z}$, is defined by

$$\Delta_a^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \sum_{s=a}^{t-\alpha} (t-s-1)^{(\alpha-1)} f(s),$$

where f is a function defined for $s = a \mod 1$ and the falling factorial power function is defined, for $\alpha \in \mathbb{R}$, as

$$t^{(\alpha)} = \begin{cases} \dfrac{\Gamma(t+1)}{\Gamma(t+1-\alpha)}, & \text{if} \quad t+1-\alpha \notin \{\dots, -3, -2, -1\}; \\ 1, & \text{if} \quad \alpha = 0; \\ 0, & \text{if} \quad t+1-\alpha \in \{\dots, -3, -2, -1\}. \end{cases}$$

Thus, $\Delta_a^{-\alpha} f$ is a function defined for $t = (a+\alpha) \mod 1$.

In particular, $\Delta_a^{-\alpha}$ maps functions defined on $\mathbb{N}_a$ to functions defined on $\mathbb{N}_{a+\alpha}$.

Remark 1. Justification of Definition 2.1 follows from discrete Cauchy formula, converting the mth multiple sum, $m \in \mathbb{N}$, into a single one

$$\sum_{k_1=a}^{t} \sum_{k_2=a}^{k_1} \cdots \sum_{k_m=a}^{k_{m-1}} f(k_m) = \frac{1}{\Gamma(m)} \sum_{k=a}^{t} (t-k+1)^{(m-1)} f(k),$$

where t, k_i and a are integers such that $a \le k_i \le k_{i-1} \le t$. Notice also that this formula plays here the analogous role of the well known Cauchy formula for iterated integrals in fractional calculus.

Another approach to Definition 2.1 can be found in [14], where the authors consider linear difference equations and the associated Green's function.

Now, after the notion of fractional sum, that of fractional difference emerges naturally and one is attempted, like in the case of continuum fractional calculus, to substitute α by $-\alpha$ in the above formulas. However, this generalization needs some care in order to guarantee a good definition which preserves well known properties of ordinary sums and differences of integer order.

Let be for this Δ^n, with $n \in \mathbb{N}$, the well known forward difference operator of order n, that is

$$\Delta^n f(s) = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} f(s+k).$$

Next, we recall two different approaches to the concept of fractional order difference operator.

Definition 2.2. Let $\alpha > 0$ such that $n-1 < \alpha \le n$ with $n = \lceil \alpha \rceil$. The fractional Caputo type difference of order α of f is defined for all $t \in \mathbb{N}_{a+\alpha}$ as, see [8, Definition 2.3],

$$^{C}\Delta_a^{\alpha} f(t) = \Delta_a^{-(n-\alpha)} \Delta^n f(t) = \frac{1}{\Gamma(n-\alpha)} \sum_{s=a}^{t-n+\alpha} (t-s-1)^{(n-\alpha-1)} \Delta^n f(s).$$