Applied Mathematics and Computation 273 (2016) 226–236

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Some inequalities for functions having Orlicz-convexity

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ARTICLE INFO

MSC:
26A33
26D15
26A51

Keywords:
Convex functions
Orlicz convexity
Fractional integrals
Hermite–Hadamard inequality

ABSTRACT

Some Hermite–Hadamard type inequalities are derived for products of functions having Orlicz-convexity properties. We also obtain these inequalities via Riemann–Liouville fractional integrals for Orlicz-convex functions. These inequalities are as best as possible from the sharpness point of view, meaning that a sharpness class of functions is identified, for each inequality, within the functions that are *s*-affine of first kind. Some special cases are discussed.

1. Introduction and preliminary results

Recently, a very general class of convexity concepts was considered by Maksa and Páles [12], where the problem of sharpness of the corresponding convexity inequality is discussed. The domain of these generalized convex functions are generalized (α, β)-convex sets of segmental type, see [2]. They are defined in [12] within a real or complex topological vector space X, by means of a nonempty set T and more functions $\alpha, \beta, a, b : T \to \mathbb{R}$. A nonempty open subset $D \subseteq X$ is said to be (α, β)-convex if $\alpha(t)x + \beta(t)y \in D$, whenever $x, y \in D$ and $t \in T$. The following functional inequality is discussed in [12] for functions $f : D \to \mathbb{R}$:

$$f(\alpha(t)x + \beta(t)y) \le a(t)f(x) + b(t)f(y), \quad \forall x, y \in D, t \in T. \tag{1}$$

The functions satisfying (1) are called (α, β, a, b)-convex and the functions that make sharp (1) are called (α, β, a, b)-affine.

The framework of the present paper is a particular case of (α, β, a, b)-convex functions, defined taking $X = \mathbb{R}$, the set of all real numbers, $D = \mathbb{R}_+ = [0, +\infty)$, $T = [0, 1]$, $s \in [0, 1]$ by the inequality

$$f(\alpha x + \beta y) \le \alpha^s f(x) + \beta^s f(y), \quad \forall x, y \in D, \alpha, \beta \in T, \alpha^s + \beta^s = 1. \tag{2}$$

This concept was introduced by Orlicz, see [17]. It is known as Orlicz-convexity but it is often named s-convexity of first kind [10] or s_1-convexity [19]. Obviously, the domain D is (α, β)-convex in this case.

First of all, we recall the main concepts. Let $\mathbb{R}$ be the set of real numbers, $I \subseteq \mathbb{R}$ an interval and $f : I \to \mathbb{R}$. Let us mention that f is a convex function, if and only if, it satisfies the inequality

$$f(tx + (1-t)y) \le tf(x) + (1-t)f(y), \ \ x, y \in I, t \in [0, 1]. \tag{3}$$

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http://dx.doi.org/10.1016/j.amc.2015.09.068
0096-3003/

The functions that fulfill the equality in (3) are called affine functions. Let $a, b \in I$ with $a < b$. Then a function f is said to be convex, if and only if, it satisfies

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)dx \le \frac{f(a)+f(b)}{2}, \tag{4}$$

which is called Hermite–Hadamard inequality, named after the first mathematicians noticing it, Hermite [9] and Hadamard [7]. Every affine function makes it sharp. The effort to prove similar inequalities accompanies every attempt to generalize the convexity concept, with more or less refined versions (see [5]).

The generalizations of the convexity concept and the cases, in which Hermite–Hadamard type inequalities occur, lead to applications in many domains. For example, useful mathematical inequalities in various generalized convexity frameworks are studied in [13,16] . Zalmai and Zhang studied minimax problems under generalized convexity assumptions in [25]. Antczak and Pitea published a parametric approach to variation under suitable convexity in [1]. The direction of generalization opened by Hanson and Mond [8] to extend the domain of convex optimization was very fruitfully modeled by Jeyakumar [11]. It generated the identification of many classes of generalized convex functions with applications in optimization. The concept of quasi-invexity, defined in [20], is used in theoretical mechanics and in nonlinear optimization by Pitea and Postolache (see [20–22]). Pitea and Antczak defined the notion of univexity and applied it to vector optimization [23]. Youness [24] opened another direction of generalization of the convexity, to model some mew nonlinear programming problems. Duca and Lupsa [6] treated this concept in epigraphic manner, as Hermite–Hadamard geometrical vision requires.

We restrict to the class of s-convexity of first kind (or in the first sense). Let us suppose that $s \in [0, 1]$. The concept introduced by the inequality (2) is equivalent to the following one, which will be used in the sequel.

Definition 1.1 ([19])**.** A function $f : I \to \mathbb{R}$ is said to be a s-convex function of first kind, also called Orlicz-convex, if

$$f\left(tx + (1-t^s)^{\frac{1}{s}}y\right) \le t^s f(x) + (1-t^s) f(y), x, y \in I, t \in [0, 1], s \in (0, 1]. \tag{5}$$

The inequality (5) is sharp for the functions called s-affine in the first sense. It is proved in [12] that, if $s \neq 1$, then the only s-affine functions in the first sense are the constants. For $s = 1$, this concept becomes the classical convexity for functions. We would like to point out that the Orlicz-convexity is a segmental type convexity property, see [2].

In Section 2 we derive some Hermite–Hadamard type inequalities for products of s-convex functions of first kind. Section 3 contains inequalities of Hermite–Hadamard type for s-convex functions of first kind which are derived via fractional integrals. In Section 4, we study the sharpness of the new inequalities.

The results derived in this paper could be useful for quantum physics, where the lower and upper bounds of natural phenomena described by integrals such as mechanical work are frequently required. For some recent results of quantum analogues of integral inequalities, see [14,15].

2. Hermite–Hadamard type inequalities for products of Orlicz-convex functions

In this section, we extend the results from [3,18] to some subclasses of Orlicz-convex functions. Some of these inequalities may be used in proving new Hermite–Hadamard type inequalities for functions having Orlicz-convexity by means of fractional integrals.

Theorem 2.1. *Let us consider two non-negative functions, $f, g : [a, b] \subset \mathbb{R}_+ \to \mathbb{R}$, and the numbers $s, p \in (0, 1]$. Let f be s-convex of first kind and g be p-convex of first kind.*

(a) *If both f and g are non-decreasing and either $f(0) = 0$ or $g(0) = 0$, then*

$$(fg)\left(\frac{a+b}{2^{\frac{1}{q}}}\right) \le \frac{1}{b-a}\int_a^b (fg)(x)dx, \tag{6}$$

with $q = min\{s, p\}$.

(b) *If both $f(0) \neq 0$ and $g(0) \neq 0$ and $f, g \in L_1[a, b]$, then*

$$f\left(\frac{a+b}{2^{\frac{1}{s}}}\right) g\left(\frac{a+b}{2^{\frac{1}{p}}}\right) \le \frac{1}{2(b-a)}\int_a^b f(x)\big[g(x) + g(a+b-x)\big]dx. \tag{7}$$

Proof.

(a) If both f and g are non-negative, non-decreasing and either $f(0) = 0$ or $g(0) = 0$, then, according to the result from [5] (first proved in [10]), the product fg is q-convex of first kind with $q = min\{s, p\}$. Then, from [5] (first proved in [4]), one gets the required inequality.

(b) Let us consider $a, b \in \mathbb{R}_+$, $a < b$, and take $x = ta + (1-t)b$, $y = (1-t)a + tb$, with $t \in [0, 1]$. Then, from (2), with $\alpha = \beta = \frac{1}{2^{\frac{1}{s}}}$, one has

$$f\left(\frac{x+y}{2^{\frac{1}{s}}}\right) \le \frac{f(x)+f(y)}{2}.$$