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Qualitative analysis of nonlinear Volterra integral equations on time scales using resolvent and Lyapunov functionals

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ABSTRACT

In this paper we use the notion of the resolvent equation and Lyapunov's method to study boundedness and integrability of the solutions of the nonlinear Volterra integral equation on time scales

$$x(t) = a(t) - \int_{t_0}^{t} C(t,s)G(s,x(s))\ \Delta s,\ \ t \in [t_0,\infty) \cap \mathbb{T}.$$

In particular, the existence of bounded solutions with various L^p properties are studied under suitable conditions on the functions involved in the above Volterra integral equation. At the end of the paper we display some examples on different time scales.

1. Introduction

The use of resolvent equation and Lyapunov‘s method have been very effective in the investigation of qualitative and quantitative properties of solutions of integral and difference equations. The resolvent equation

$$r(t,s) = -a(t,s) + \int_{s}^{t} r(t,u)a(u,s)ds$$

for the integral equation

$$x(t) = f(t) + \int_{0}^{t} a(t,s)x(s)ds$$

is constructed in [25]. There are many papers and books written on the subject of qualitative and quantitative properties of integral equations. The literature on traditional integral equations is vast and we only mention [10,13–18,21]. For most recent and prominent results we direct the reader to [14] and [21]. For discrete Volterra equations we refer the interested reader to [3,16,17,27] and [28]. The use of Lyapunov functionals on time scale is being rapidly developed. The articles [2,20], and [23] deal with general theory in which the existence of such a Lyapunov functional is assumed, so that existence and stability results can be inferred. In [24] the authors used fixed point theory and obtained sufficient conditions guaranteeing the existence and uniqueness of solutions of Volterra integral equations on time scales.

The existence of resolvent on time scales for integral equations was initially developed and proved in [1]. The results of this paper complement [1], where for the first time the resolvent is being used to qualitatively study the solutions of integral

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equations on time scales. Moreover, in Section 3, the authors display a Lyapunov functional on time scales and obtain various results regarding the L^p solutions of (1.2). Thus, this research will serve as the corner stone of future studies on time scales whether using Lyapunov functional or the notion of resolvent.

Constructing Lyapunov functionals for integral equations has been very challenging until recently, even in the continuous case. Burton (see e.g. [12]) was the first one to construct such functions and utilize them to qualitatively analyze solutions of integral equations. The study of integral equations on time scales provides deeper and comprehensive understanding of traditional integral equations and summation equations. Some of the results in this paper are new for the continuous case and all of them are new for the discrete case.

We begin by stating some important facts and properties of time scales that we will be using during our analysis.

A time scale, denoted $\mathbb{T}$, is a nonempty closed subset of real numbers. The set $\mathbb{T}^\kappa$ is derived from the time scale $\mathbb{T}$ as follows: if $\mathbb{T}$ has a left-scattered maximum M, then $\mathbb{T}^\kappa = \mathbb{T} - \{M\}$, otherwise $\mathbb{T}^\kappa = \mathbb{T}$. The delta derivative f^Δ of a function $f : \mathbb{T} \to \mathbb{R}$, defined at a point $t \in \mathbb{T}^\kappa$ by

$$f^\Delta(t) := \lim_{s\to t} \frac{f(\sigma(t)) - f(s)}{\sigma(t) - s}, \quad \text{where } s \to t, \ s \in \mathbb{T}\backslash\{\sigma(t)\}, \tag{1.1}$$

was first introduced by Hilger [19] to unify discrete and continuous analyses. In (1.1), $\sigma : \mathbb{T} \to \mathbb{T}$ is the forward jump operator defined by $\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}$. Hereafter, we denote by $\mu(t)$ the step size function $\mu : \mathbb{T} \to \mathbb{R}$ defined by $\mu(t) := \sigma(t) - t$. A point $t \in \mathbb{T}$ is said to be right dense (right scattered) if $\mu(t) = 0$ ($\mu(t) > 0$). A point is said to be left dense if $\sup\{s \in \mathbb{T} : s < t\} = t$. A function $f : \mathbb{T} \to \mathbb{R}$ is called *rd*-continuous if it is continuous at right dense points and its left sided limits exist (finite) at left dense points. We denote by $C_{rd}(\mathbb{T}, \mathbb{R})$ the set of all *rd*-continuous functions. Every *rd*-continuous function $f : \mathbb{T} \to \mathbb{R}$ has an anti-derivative F denoted by

$$F(t) := \int_{t_0}^t f(t)\Delta t.$$

For a comprehensive review on delta derivative and delta integral on time scales we refer to [6–9]. We assume the reader is familiar with the calculus of time scales.

Throughout the paper, we suppose that $\mathbb{T}$ is a time scale that is unbounded above and denote by $L^p(I_\mathbb{T})$ the Banach space of real valued functions on the interval $I_\mathbb{T} := [t_0, \infty) \cap \mathbb{T}$ endowed by the norm

$$\|f\|_p := \left(\int_{t\in I_\mathbb{T}} |f(t)|^p \Delta t\right)^{1/p}$$

(see [26]).

In Section 2, we will use a variant form of the resolvent equation that was developed in [1] and use fixed point theory to obtain boundedness of the solutions of the nonlinear Volterra integral equation on time scales,

$$x(t) = a(t) - \int_{t_0}^t C(t,s)G(s,x(s))\Delta s, \ t \in I_\mathbb{T}, \tag{1.2}$$

where $t_0 \in \mathbb{T}^\kappa$ is fixed and the functions $a : I_\mathbb{T} \to \mathbb{R}$, $C : I_\mathbb{T} \times I_\mathbb{T} \to \mathbb{R}$, and $G : I_\mathbb{T} \times \mathbb{R} \to \mathbb{R}$ are supposed to be continuous. Based on the results of [1], we have the following. Given a linear system of integral equations of the form

$$x(t) = a(t) - \int_{t_0}^t C(t,s)x(s)\Delta s, \quad t_0 \in \mathbb{T}^\kappa \tag{1.3}$$

the corresponding resolvent equation associated with $C(t, s)$ is given by

$$R(t,s) = C(t,s) - \int_{\sigma(s)}^t R(t,u)C(u,s)\Delta u. \tag{1.4}$$

If $C(t, s)$ is scalar valued, then so is $R(t, s)$. If $C(t, s)$ is $n \times n$ matrix, then so is $R(t, s)$. Moreover, the solution of (1.3) in terms of R is given by

$$x(t) = a(t) - \int_{t_0}^t R(t,u)a(u)\Delta u. \tag{1.5}$$

2. L^∞ solutions: Resolvent and Fixed Point methods

We limit our discussion to scalar integral equations. It will be easy to prove similar theorems for vector systems. We first begin with following lemma which enables us to change the order of integration on a triangular region.

Lemma 1 (Change of order of integration)**.** *[22, Theorem 3.2] Let $\varphi \in C_{rd}(\mathbb{T} \times \mathbb{T}, \mathbb{R})$ be an rd-continuous function. For $t, t_0 \in \mathbb{T}^\kappa$ we have*

$$\int_{t_0}^t \int_{\sigma(s)}^t \varphi(u,s)\Delta u\Delta s = \int_{t_0}^t \int_{t_0}^u \varphi(u,s)\Delta s\Delta u. \tag{2.1}$$