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Extremal Laplacian energy of threshold graphs

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ABSTRACT

Let G be a connected threshold graph of order n with m edges and trace T. In this paper we give a lower bound on Laplacian energy in terms of n, m and T of G. From this we determine the threshold graphs with the first four minimal Laplacian energies. Moreover, we obtain the threshold graphs with the largest and the second largest Laplacian energies.

1. Introduction

In this paper, we deal with connected threshold graphs G with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G)$, where $|E(G)| = m$. Let d_i be the degree of vertex v_i for $i = 1, 2, \ldots, n$ such that $d_1 \geq d_2 \geq \cdots \geq d_n$. The maximum vertex degree is denoted by $\Delta = \Delta(G)$ in G. The Laplacian matrix of G is given by $L(G) = D(G) - A(G)$, where $D(G)$ is the diagonal matrix of vertex degrees and $A(G)$ is the adjacency matrix of G. Denote by $Spec(G) = \{\mu_1, \mu_2, \ldots, \mu_n = 0\}$ the spectrum of $L(G)$, i.e., the Laplacian spectrum of G, where $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_{n-1} \geq \mu_n = 0$. It is well known that the Laplacian eigenvalues are integers for all threshold graphs. The Laplacian energy of the graph G is defined as [15]

$$LE = LE(G) = \sum_{i=1}^{n} \left|\mu_i - \frac{2m}{n}\right|. \tag{1}$$

For its basic mathematical properties, including various lower and upper bounds, see [9,11,30]. Li et al. obtained a relation between Laplacian energy and Laplacian Estrada index of graphs [22]. It is worth noting that LE found applications not only in theoretical organic chemistry (see, [29]), but also in image processing [31] and information theory [19]. In the literature several other "graph energies" have been considered. The oldest and most thoroughly studied is adjacency energy [10,14,17,19–21,29] and some other are the Randić energy [1,12], incidence energy [2,3,6], Laplacian-energy-like invariant [7,25], distance energy [24,32] and matching energy [5,18,23,34].

A Ferrers–Sylvester diagram (see, Fig. 2) is a grid representing a degree sequence $(d) = (d_1, d_2, \ldots, d_n)$ in which the ith row of the grid contains d_i boxes. The conjugate of a degree sequence (d) is the sequence $(d^*) = (d_1^*, d_2^*, \ldots, d_k^*)$ where $d_i^* = |\{j : d_j \geq i\}|$. Visually speaking, the value for d_i^* is the number of boxes in the ith column of the Ferrers–Sylvester diagram. The trace denoted by T, of a degree sequence $d = (d_1, d_2, \ldots, d_n)$ is the largest integer i such that $d_i \geq i$ [28], that is,

$$T = \max\{i : d_i \geq i\}. \tag{2}$$

A threshold graph is obtained through an iterative process which starts with an isolated vertex, and at each step, either a new isolated vertex is added, or a vertex adjacent to all previous vertices (dominating vertex) is added. From the definition of threshold graph we have $d_{T+1} = T$.

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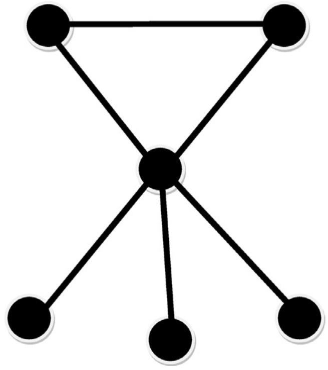

Fig. 1. Graph S_n^1 with trace $T = 2$.

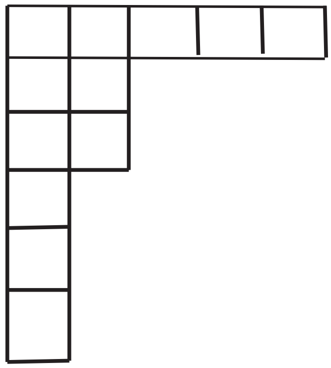

Fig. 2. Ferrers–Sylvester diagram of S_n^1.

Vinagre et al. [33] found maximum Laplacian energy among threshold graphs with condition $\mu_T \geq \frac{2m}{n} \geq \mu_{T+1}$. Helmberg et al. [16] found a threshold graph with maximal Laplacian energy. As usual, K_n and S_n, denote, respectively, the complete graph and the star on n vertices. Denote by S_n^1 the graph obtained by adding a new edge between two pendent vertices of the star S_n of order n (see Fig. 1).

The paper is organized as follows. In Section 2, we give a list of some previously known results. In Section 3, we present a lower bound on Laplacian energy in terms of n, m and trace T of threshold graph G. From this we obtain the threshold graphs with the first four minimal Laplacian energies. In Section 4, we determine the threshold graphs with the largest and the second largest Laplacian energies.

2. Preliminaries

In this section, we shall list some previously known results that will be needed in the next two sections.

Lemma 2.1. *[27] Let G be a graph with degree sequence* (d) *and trace T. Then G is a threshold graph if and only if* $d_i + 1 = d_i^*$, $1 \leq i \leq T$.

Lemma 2.2. *[27] Let G be a threshold graph with degree sequence* (d) *and trace T. Then* $d_{i+1} = d_i^*$, $i > T$.

Lemma 2.3. *[27] Let G be a threshold graph on n vertices with conjugate degree sequence* $(d^*) = (d_1^*, d_2^*, \ldots, d_n^*)$. *Then* $\mu_i = d_i^*$ *for* $1 \leq i \leq n-1$.

Let σ $(1 \leq \sigma \leq n-1)$ be the largest positive integer such that

$$\mu_\sigma \geq \frac{2m}{n}. \tag{3}$$

Then from [8], we have