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Stability analysis of time-delay systems using a contour integral method

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ABSTRACT

In this paper we propose a contour integral method for computing the rightmost characteristic roots of systems of linear time-delay differential equations (DDEs). These roots are very important in the context of stability analysis of the time-delay systems. The effectiveness of the proposed method is illustrated by some numerical experiments.

1. Introduction

In this paper we consider the (asymptotic) stability of a system of linear time-delay differential equations (DDEs),

$$\dot{y}(t) = A_0 y(t) + \sum_{i=1}^{k} A_i y(t - \tau_i), \tag{1.1}$$

where $y(t) \in R^n$ is the state variable at time t, $A_i \in R^{n\times n}(i = 0, 1, \dots, k)$, and $\tau_i \geq 0(i = 1, 2, \dots, k)$ represent the time-delays.

The characteristic roots of the system (1.1) can be computed [1] as the eigenvalues of the following nonlinear eigenvalue problem (NEP)

$$T(\lambda)x = 0, \tag{1.2}$$

where $\lambda \in C, x \in C^n(x \neq 0)$ and $T(\lambda)$ is the following characteristic matrix

$$T(\lambda) = \lambda I - A_0 - \sum_{i=1}^{k} A_i e^{-\lambda \tau_i}. \tag{1.3}$$

As we know, the system (1.1) is (asymptotically) stable if all eigenvalues of the NEP (1.2) have (strict) negative real parts, see, e.g., [2]. Note that there are an infinite number of eigenvalues of the NEP (1.2), however, only a finite number of eigenvalues for the NEP (1.2) have real parts greater than zero, see [3] for details. Hence, the stability of the system (1.1) depends on these finite number of eigenvalues. For these eigenvalues we must find an eigenvalue whose real part is biggest and call this eigenvalue as the rightmost eigenvalue of the NEP (1.2). Therefore, a numerical method for computing the rightmost eigenvalues of the NEP (1.2) would be of interest.

At present, there are mainly two types of numerical methods to find the rightmost eigenvalues of the NEP (1.2). The first type of numerical methods is based on the discretization of the solution operator associated with the system (1.1) using the linear

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multi-step (LMS) time-integration method, see, e.g., [4–6]. The second type of numerical methods is based on the discretization of the infinitesimal generator of the solution operator via a spectral method, see, e.g., [7–10]. However, both two types of numerical methods transfer the problem of computing the characteristic roots of the finite-dimensional nonlinear eigenvalue problem (1.2) to the solutions of the infinite-dimensional linear eigenvalue problem.

In addition, individual eigenvalues including the rightmost eigenvalues can be found efficiently using Newton's methods with a suitable starting value, see, e.g., [11–15]. However, these methods are locally and quadratically convergent, and cannot be guaranteed to compute all eigenvalues in a given region. To overcome the disadvantage of the Newton methods, a contour integral method is proposed to compute all eigenvalues of the NEP (1.2) in a specific region of the complex plane. For example, Sakurai et al. [16–18] presented a quadrature method for solving the generalized eigenvalue problems (GEPs) and gave its error analysis. Sakurai et al. [19,20] developed a Rayleigh–Ritz type method for the GEPs with contour integral. Asakura et al. [21] proposed a contour integral method for polynomial eigenvalue problems (PEPs). Asakura et al. [22], Beyn [23] and Yokota and Sakurai [24] presented a contour integral method to solve the NEP (1.2). As far as we know the contour integral method has not been applied to analyze the stability of time-delay systems until now. In this paper, we establish the relation between the stability of the system (1.1) and the rightmost eigenvalues of the NEP (1.2) and propose a novel method to compute the rightmost eigenvalues of the NEP (1.2) for the stability of the system (1.1).

The remaining of the paper is organized as follows. In Section 2, we present a contour integral method for computing all eigenvalues of the NEP (1.2) in a specific region. In Section 3, we propose a novel method to find the rightmost eigenvalues of the NEP (1.2) using the contour integral method. In Section 4, some numerical experiments are given to show the effectiveness of the proposed method. Finally, we conclude in Section 5.

For convenience, we use the following notations: $det(A)$ and $tr(A)$ denote the trace and the determinant of a matrix A, respectively; $\lambda(A)$ denotes all the eigenvalues of a matrix A; ι denotes the imaginary unit of a complex number; $\|A\|$ denotes 1-norm, or 2-norm or ∞-norm of a matrix A. Γ is the boundary of an open disk Ω, its center and radius are γ and ρ, respectively.

2. Computing all eigenvalues in an open disk by the contour integral method

For the NEP (1.2), we define a function $f(\lambda) = det(T(\lambda))$, then $f(\lambda)$ is an analytic function of λ. It is well known that λ is an eigenvalue of the NEP (1.2) if and only if $f(\lambda) = 0$.

We introduce briefly the contour integral method to compute all eigenvalues of the NEP (1.2) in the open disk Ω, see [22–25] for details.

Let $m (m \le n)$ be a positive integer and the complex moments υ_p be

$$\upsilon_p = \frac{1}{2\pi\iota}\oint_{\Gamma}(z-\gamma)^p\frac{f'(z)}{f(z)}dz, \quad p = 0, 1, \ldots, 2m-1. \tag{2.1}$$

Using $\upsilon_p (p = 0, 1, \ldots, 2m-1)$, we define two $m \times m$ Hankel matrices H_m and $H_m^<$ as follows

$$H_m = \begin{pmatrix} \upsilon_0 & \upsilon_1 & \cdots & \upsilon_{m-1} \\ \upsilon_1 & \upsilon_2 & \cdots & \upsilon_m \\ \vdots & \vdots & & \vdots \\ \upsilon_{m-1} & \upsilon_m & \cdots & \upsilon_{2m-2} \end{pmatrix}$$

and

$$H_m^< = \begin{pmatrix} \upsilon_1 & \upsilon_2 & \cdots & \upsilon_m \\ \upsilon_2 & \upsilon_3 & \cdots & \upsilon_{m+1} \\ \vdots & \vdots & & \vdots \\ \upsilon_m & \upsilon_{m+1} & \cdots & \upsilon_{2m-1} \end{pmatrix}.$$

Then we can compute all the eigenvalues $\lambda_i (i = 1, 2, \ldots, m)$ of the GEP $H_m^< x = \lambda H_m x$.

Theorem 2.1 [25]. *Assume that $f(\lambda)$ has m different zeros $\lambda_1, \lambda_2, \ldots, \lambda_m$ inside Γ. Then $\lambda_i - \gamma (i = 1, 2, \ldots, m)$ are the eigenvalues of the generalized eigenvalue problem $H_m^< x = \lambda H_m x$ if and only if $\lambda_i (i = 1, 2, \ldots, m)$ are the zeros of $f(\lambda)$ inside Γ.*

Now, we consider that the integration is evaluated via a trapezoidal rule on the circle Γ. Let N be the number of sample points on the circle Γ and $\omega_j = \gamma + \rho \exp(2\pi j\iota/N) (j = 0, 1, \ldots, N-1)$. Based on the trapezoidal rule, the complex moments υ_p can be approximately obtained by following formula

$$\upsilon_p \approx \widehat{\upsilon}_p = \frac{1}{N}\sum_{j=0}^{N-1}(\omega_j - \gamma)^{p+1}\frac{f'(\omega_j)}{f(\omega_j)}.$$

It follows from the Trace–Theorem of Devidenko (see [26]) $\frac{f'(\lambda)}{f(\lambda)} = tr(T^{-1}(\lambda)T'(\lambda))$ that $\widehat{\upsilon}_p$ can be rewritten as

$$\widehat{\upsilon}_p = \frac{1}{N}\sum_{j=0}^{N-1}(\omega_j - \gamma)^{p+1} tr(T^{-1}(\omega_j)T'(\omega_j)). \tag{2.2}$$