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A perturbative algorithm for quasi-periodic linear systems close to constant coefficients

Ana Arnal, Fernando Casas*, Cristina Chiralt

Institut de Matemàtiques i Aplicacions de Castelló (IMAC) and Departament de Matemàtiques, Universitat Jaume I, Castellón E-12071, Spain

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ABSTRACT

A perturbative procedure is proposed to formally construct analytic solutions for a linear differential equation with quasi-periodic but close to constant coefficients. The scheme constructs the necessary linear transformations involved in the reduction process up to an arbitrary order in the perturbation parameter. It is recursive, can be implemented in any symbolic algebra package and leads to accurate analytic approximations sharing with the exact solution important qualitative properties. This algorithm can be used, in particular, to carry out systematic stability analyses in the parameter space of a given system by considering variational equations.

1. Introduction

In this paper we consider the differential equation

$$\dot{y} \equiv \frac{dy}{dt} = (A_0 + Q(t, \varepsilon))y, \tag{1}$$

where $y \in \mathbb{C}^d$, $\varepsilon > 0$, A_0 is a constant $d \times d$ matrix and

$$Q(t, \varepsilon) \equiv \sum_{j\geq 1} \varepsilon^j A_j(t) \tag{2}$$

is a quasi-periodic $d \times d$ matrix function of t with frequencies $(\omega_1, \ldots, \omega_r)$.

We recall that a function f is said to be *quasi-periodic* with basic frequencies $\omega = (\omega_1, \ldots, \omega_r)$ if $f(t) = F(\theta_1, \ldots, \theta_r)$, where F is 2π-periodic with respect to $\theta_1, \ldots, \theta_r$ and $\theta_j = \omega_j t$ for $j = 1, \ldots, r$. Quasi-periodic functions have a representation of the form

$$f = \sum_{k\in\mathbb{Z}^r} f_k \, e^{i(k,\omega)t}$$

where $(k, \omega) \equiv k_1\omega_1 + \cdots + k_r\omega_r$ and $\Sigma |f_k|^2 < \infty$ [10].

In connection with system (1)–(2), the issue of reducibility has received much attention along the years. Roughly speaking, Eq. (1) is said to be reducible if there exists a change of variables $y = P(t)z$ defined by a nonsingular quasi-periodic and continuously differentiable matrix $P(t)$ such that z satisfies the equation $\dot{z} = Kz$, with K a constant matrix. In the purely periodic case,

* Corresponding author. Tel.: +34 964728374.
E-mail addresses: ana.arnal@uji.es (A. Arnal), Fernando.Casas@uji.es (F. Casas), chiralt@uji.es (C. Chiralt).

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$A(t+T,\varepsilon)=A(t,\varepsilon)$, with $T>0$, the well known Floquet theorem [24] guarantees reducibility by means of a periodic transformation $P(t)$ with the same period T. Moreover, every fundamental matrix solution of (1) can be written globally as

$$Y(t)=P(t)\,\mathrm{e}^{Kt}. \tag{3}$$

In the more general case of a quasi-periodic system which is close to constant coefficients, i.e. Eq. (2) with sufficiently small ε, the analysis is more involved. At least two different strategies can be found in the literature. The first one, proposed by Shtokalo [23] and later analyzed by several authors [11,13,21], consists in formally constructing a change of variables

$$y=P(t,\varepsilon)\,z=\left(I+\sum_{n=1}^{\infty}\varepsilon^{n}P_{n}(t)\right)z, \tag{4}$$

as a power series in ε, so that Eq. (1) with (2) is transformed into

$$\dot{z}=K(\varepsilon)z\equiv\left(A_0+\sum_{j=1}^{\infty}\varepsilon^{j}K_j\right)z, \tag{5}$$

where K_j are constant matrices. Recursive procedures exist to compute the quasi-periodic matrices $P_n(t)$ and the constant terms K_j at each iteration (see e.g. [6]). Essentially, the K's are determined by averaging and subsequently the P's are obtained by solving the corresponding differential equation. Although the procedure only allows one to construct asymptotic expansions for the solution, it is still possible to provide sufficient conditions guaranteeing stability or instability of the trivial solution of (1) from the solution $z=0$ of system (5) once truncated at the, say, sth-iteration [6,21]. Moreover, the technique can be generalized to analyze the asymptotic behavior of linear differential equations with oscillatory decreasing coefficients [6,18].

The second approach is very much in the spirit of the proof of KAM theorem in Hamiltonian systems, as given, for instance, in [1], and was first considered in [3]: instead of just one change of variables (4), a sequence of successive quasi-periodic linear transformations is constructed with the aim not of eliminating all powers of ε at once, but to increase in each step the order of the perturbation by the square of the preceding one. In this way, it is possible to establish rigorous results on the reducibility of (1) when ε is small [3,14,15]. More specifically, let λ_i denote the eigenvalues of A_0 and let $\alpha_{ij}=\lambda_i-\lambda_j$ for $i\neq j$. Then if all values $\mathrm{Re}\,\alpha_{ij}\neq 0$ the system is reducible for $|\varepsilon|<\varepsilon_0$, ε_0 sufficiently small [3], whereas if some of the $\mathrm{Re}\,\alpha_{ij}$ are zero (as happens when the λ_i are purely imaginary) more hypotheses are required. Thus, if the α_{ij} and the basic frequencies of $Q(t,\varepsilon)$ satisfy diophantine (non resonant) conditions and a certain non degeneracy holds with respect to ε, then there exists a Cantorian set of positive measure $\mathcal{E}$ such that for $\varepsilon\in\mathcal{E}$ system (1) is reducible by means of a quasi-periodic change of variables [15]. In other words, if the parameter ε is small enough, reducibility can be achieved only for a set of values of ε with empty interior but large Lebesgue measure (maybe full measure). On the other hand, just by assuming a non resonant condition of α_{ij} and the basic frequencies of $Q(t,\varepsilon)$ it is possible to transform the original system (1) by means of a sequence of quasi-periodic matrices to

$$\dot{z}=(K(\varepsilon)+\varepsilon Q^{*}(t,\varepsilon))z,\qquad |\varepsilon|\leq\varepsilon_0 \tag{6}$$

where Q^* is exponentially small in ε [14]. Equivalently, instead of aiming at a total reduction of system (1), the goal is to minimize the quasi-periodic part (up to exponentially small terms) without taking out any value of ε. In addition, there is no need to impose further non degeneracy conditions [14]. In fact, the procedure developed in [14] allowed the authors to compute numerically for a 2×2 system the change of variables in such a way that, for a given (small) value of ε, the size of the remainder Q^* in (6) is kept below a certain predefined tolerance.

Our purpose in this paper, rather than analyzing conditions guaranteeing total or partial reducibility of system (1), consists in devising an algorithm for constructing the necessary linear transformations involved in the reduction process in such a way that (i) it is computationally well adapted so that the analytic procedure can be carried out at high orders in ε and (ii) the "effective" Floquet factorization that results from the corresponding approximations to the transformation $P(t,\varepsilon)$ and the matrix $K(\varepsilon)$ allows us to construct analytic approximate expressions for the fundamental matrix of (1) in a way that other qualitative properties of the exact solution (e.g., symplecticity or unitarity) are exactly preserved. This is so when only one linear change of variables is involved, as in (4), but also when a sequence of transformations is considered. The algorithm constitutes a generalization of that presented in [8] for periodic systems and consists essentially in constructing $P(t,\varepsilon)$ as a matrix exponential whose generator $L(t,\varepsilon)$ satisfies a cohomological equation at each order. The analytic approximations thus obtained are in addition free of secular terms.

We are well aware that, even when the system is reducible for a given value of ε, the resulting transformation P might be far from the identity. This in fact is quite common for "moderate" values of ε [14,15,20]. Under such circumstances, our construction would be purely formal, of course. Nevertheless, as the examples collected in the paper show, the procedure is still able to provide reasonably accurate results even in this situation.

2. The general algorithm

2.1. One transformation

The algorithm we use to construct the approximations can be considered as a generalization of the procedure presented in [8]. For the benefit of the reader, we collect here only the main points of this procedure and refer to [8] for a more detailed treatment. We start with the case of only one transformation.