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An adaptive coupling method for exterior anisotropic elliptic problems☆

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A R T I C L E I N F O

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A B S T R A C T

In this paper, we propose an adaptive coupling method for solving anisotropic elliptic PDEs in unbounded domains. Firstly, the existence and the uniqueness of the solution for the coupling method are proven, and the a priori error estimates in H^1-norm and L^2-norm that depend on the size of the FEM mesh, the location of the elliptic artificial boundary and the truncation of the infinite series in the artificial boundary integral condition are derived. Secondly, the a posteriori error estimates and the a posteriori error indicator of the coupling method are obtained. Finally, the adaptive coupling method refines the mesh distribution by the arc-length equidistribution principle and the a posteriori error indicator successively. Numerical examples confirm the advantage in accuracy and efficiency for the proposed method.

1. Introduction

Many practical applications on acoustics, aerodynamics, hydrodynamics, electromagnetism, geophysics, meteorology, environmental science, etc., often come down to the problems of partial differential equations in unbounded domains. But the unboundedness of the domains brings about the essential difficulty for solving these problems. The existing mature numerical methods, such as the finite difference method and the finite element method cannot be directly applied in the solution of such problems. Usually, the problems in infinite regions need to be properly transformed into the problems in finite regions, and then to be solved [1–6]. As well-known, the artificial boundary method converts the original exterior problem into an equivalent or an approximate problem in the bounded computational region by introducing an artificial boundary and constructing some accurate (or high-precision) boundary condition on it. The natural boundary integral equation or said DtN (Dirichlet to Neumann) mapping as an important artificial boundary condition is favored by many scholars, because it is fully compatible with the finite element method [3,7–9].

For the exterior Dirichlet problem of two-dimensional Laplace equation with the boundary Γ in $\Re^2$, set the artificial boundary Γ_e to be a circle with radius R to enclose Γ, denote Ω_i to be the computational region between Γ and Γ_e, and denote N to be the number of terms of the truncated series in the natural boundary integral equation. Han [10] obtained an error estimate $\|u-u_h^N\|_{1,\Omega_i} \le C(h\|u\|_{2,\Omega_i} + \frac{1}{N^{k-1}}\|u\|_{k-1/2,\Gamma_e})$ by using the Fourier analysis for the DtN-FEM solution. Yu [11] obtained an error estimate that depended on N, h, R/R_0 and u, where R_0 was the radius of a concentric circle of Γ_e to enclose Γ. Similar error estimates were generalized for solving the exterior problems in the unbounded strip area and the cylinder [12–14] and exterior

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problems of Helmholtz equation [15,16]. The exterior problems of Laplace equation and Helmholtz equation were also solved by using the elliptic and ellipsoidal artificial boundaries [17–19].

Recently, the Laplace equation in the upper half-plane was solved by using the wavelet collocation method in [20]. A coupled finite-infinite element method for the exterior Poisson problem was employed and its error estimate was derived in [21]. A Schwarz alternating algorithm was investigated for a three-dimensional exterior harmonic problem with prolate spheroid boundary in [22]. Two a-posteriori error analyses were developed for the dual-mixed finite element method with a Dirichlet-to-Neumann mapping to solve linear exterior transmission problems in the plane in [23]. By using the Kirchhoff transformation and the Fourier series expansion, the exact artificial boundary condition was established on a circular or spherical artificial boundary to solve quasilinear elliptic problems [24,25]. The readers who are interested in the artificial boundary method to solve more linear and nonlinear of PDEs on unbounded domains are recommended to see surveys [7,26] and books [6,8,9].

In this study, we discuss not only the effect of DtN boundary condition but also the application of adaptive methods in DtN-FEM. Three main kinds of adaptive methods are as follows: the p-adaptive, the h-adaptive and the r-adaptive. Being different from the p-adaptive and the h-adaptive, the r-adaptive neither improves the degree of the approximating polynomial nor increases the total number of grid points, it only changes the positions of the grid points to implement the adaptivity. In the process of solving problems, the r-adaptive method can keep the topological structure among grid points fixed, adjust the positions of the grid points by redistributing them to equidistribute the monitor function in order to reduce the global error, and improve the accuracy of numerical solutions [27–29]. However, the r-adaptive cannot make the algorithm convergence by itself. The h-adaptive was successfully applied in the natural BEM and the DtN-FEM based on the a posterior error estimate [30–33]. So, in order to achieve better effect, we need to combine the h-adaptive [34,35] and the r-adaptive [27,28,36,37].

In this paper, we propose an adaptive coupling method for solving the anisotropic elliptic partial differential equation in 2-D unbounded domains. In Section 2, we prove a priori error estimates of second-order in $L^2(\Omega_i)$ from a priori error estimates of first-order in $H^1(\Omega_i)$ for the coupling method. In Section 3, we derive two a posteriori error estimates and an error indicator for the h-adaptive coupling method. In Section 4, besides an hr-adaptive algorithm, the advantage in accuracy and efficiency is illustrated, and the correctness of the related error estimates is confirmed by numerical examples.

2. The coupling method and its a priori error estimates

Consider to solve the exterior Dirichlet boundary value problem of the anisotropic elliptic partial differential equation:

$$\begin{cases} -a\dfrac{\partial^2 u}{\partial x^2} - b\dfrac{\partial^2 u}{\partial y^2} = f, & \text{in } \Omega, \\ u = g, & \text{on } \Gamma, \end{cases} \tag{1}$$

where Γ is a polygonal boundary surrounding the origin in $\mathcal{R}^2$, Ω is the unbounded domain outside Γ, $f \in H^{-1}(\Omega)$ has compact support, $g \in H^{\frac{1}{2}}(\Gamma)$, and u is bounded at infinity.

For a long and narrow boundary Γ or as a general case, by making an elliptic artificial boundary $\Gamma_e = \{(x, y)|\alpha x^2 + \beta y^2 = 1\}$ to enclose Γ and the support of f, the Ω is divided into two subdomains, namely the bounded subdomain Ω_i and the unbounded subdomain Ω_e. By the variable transform $x = \sqrt{a}\xi, y = \sqrt{b}\eta$, the problem (1) can be turned into:

$$\begin{cases} -\dfrac{\partial^2 u}{\partial \xi^2} - \dfrac{\partial^2 u}{\partial \eta^2} = f, & \text{in } \Omega, \\ u = g, & \text{on } \Gamma. \end{cases} \tag{2}$$

By introducing elliptic coordinates $\xi = f_0 \cosh\mu_0 \cos\varphi$, $\eta = f_0 \sinh\mu_0 \sin\varphi$, where $f_0 = \sqrt{\frac{b\beta - a\alpha}{a\alpha b\beta}}$ and $\mu_0 = \ln\frac{\sqrt{b\beta}+\sqrt{a\alpha}}{\sqrt{b\beta - a\alpha}}$, and by Fourier series expansion on $\Gamma_e = \{(\xi, \eta)|a\alpha\xi^2 + b\beta\eta^2 = 1\} = \{(\mu, \theta)|\mu = \mu_0, 0 \le \theta \le 2\pi\}$, we have the Poisson formula:

$$u(\mu, \varphi) = \frac{e^{2\mu} - e^{2\mu_0}}{2\pi}\int_0^{2\pi} \frac{u(\mu_0, \varphi')}{e^{2\mu} + e^{2\mu_0} - 2e^{\mu+\mu_0}\cos(\varphi - \varphi')}\mathrm{d}\varphi', \quad \mu > \mu_0, \tag{3}$$

and the natural boundary integral equation:

$$\mathcal{K}u = -\frac{1}{\sqrt{J}}\int_0^{2\pi} \frac{1}{4\pi \sin^2(\frac{\varphi-\varphi'}{2})}u(\mu_0, \varphi')\mathrm{d}\varphi', \tag{4}$$

where $J = f_0^2 \sinh^2\mu_0 \cos^2\varphi + f_0^2 \cosh^2\mu_0 \sin^2\varphi$ (see [28]). The natural boundary integral equation (4) is a relation between Neumann data and Dirichlet data. $\mathcal{K}$ is a hypersingular integral operator, i.e. the DtN operator.

By the important formula (see [8]):

$$-\frac{1}{4\pi \sin^2(\frac{\varphi}{2})} = \sum_{m=-\infty}^{+\infty} \frac{1}{2\pi}|m|e^{im\varphi} = \frac{1}{\pi}\sum_{m=1}^{+\infty} m\cos m\varphi, \tag{5}$$