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Lie symmetries for Lie systems: Applications to systems of ODEs and PDEs

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ABSTRACT

A *Lie system* is a nonautonomous system of first-order differential equations admitting a *superposition rule*, i.e., a map expressing its general solution in terms of a generic family of particular solutions and some constants. Using that a Lie system can be considered as a curve in a finite-dimensional Lie algebra of vector fields, a so-called *Vessiot–Guldberg Lie algebra*, we associate every Lie system with a Lie algebra of Lie point symmetries induced by the Vessiot–Guldberg Lie algebra. This enables us to derive Lie symmetries of relevant physical systems described by first- and higher-order systems of differential equations by means of Lie systems in an easier way than by standard methods. A generalization of our results to partial differential equations is introduced. Among other applications, Lie symmetries for several new and known generalizations of the real Riccati equation are studied.

1. Introduction

The analysis of Lie systems dates back to the end of the XIXth century, when Vessiot, Guldberg and Lie [22,25,33] discovered the most fundamental properties of nonautonomous systems of first-order ordinary differential equations (ODEs) admitting a superposition rule. Lie proved the nowadays called *Lie–Scheffers Theorem*, which states that a nonautonomous system of first-order ODEs admits a superposition rule if and only if it describes the integral curves of a t-dependent vector field taking values in a finite-dimensional real Lie algebra V of vector fields [9]. We call V a *Vessiot–Guldberg Lie algebra* of the Lie system [12].

The theory of Lie systems was deeply analyzed towards the end of the XIXth century, to be almost forgotten soon after. Along the eighties of the XXth century, Winternitz and coworkers derived superposition rules for many relevant systems of differential and superdifferential equations and studied the classification of Lie systems on manifolds among other topics [4,35].

Cariñena, Grabowski and Marmo characterized superposition rules as connections of zero curvature [9]. Other relevant results have also been derived for Lie systems admitting a Vessiot–Guldberg Lie algebra of Hamiltonian vector fields with respect to some geometric structure [7,13]. This fact led to the derivation of a superposition rule for Riccati equations in a simpler way by using a Casimir function of $C^\infty(\mathfrak{sl}(2,\mathbb{R})^*)$ [3]. These results illustrate that Lie systems enjoy relevant geometric properties. Additionally, Lie systems can be applied in quantum/classical mechanics and control theory (see [12] and references therein).

The interest of Lie symmetries for the study of systems of differential equations is undoubtable. For instance, Lie symmetries allow us to write a differential equation in a simpler way. In this paper, we pioneer the study of Lie symmetries for Lie systems

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and their generalizations to higher-order ODEs (HODEs) and partial differential equations (PDEs): the *HODE and PDE Lie systems* [9,10]. As far as we know, only a few minor results about Lie symmetries for Lie systems had appeared before in [13].

Given a Lie system possessing a Vessiot–Guldberg Lie algebra V, we study its Lie symmetries of the form

$$Y = f_0(t)\frac{\partial}{\partial t} + \sum_{\alpha=1}^{r} f_\alpha(t)X_\alpha,$$

where $f_0, \ldots, f_r$ are certain t-dependent functions and $X_1, \ldots, X_r$ form a basis for V. We prove that $f_0, \ldots, f_r$ depend on the algebraic structure of V and can be derived by solving a Lie system, the symmetry system of the Lie system under study, which can be expressed in coordinates in a simple way. This enables us to calculate Lie symmetries for Lie systems with isomorphic Vessiot–Guldberg Lie algebras simultaneously.

As an application, we study Lie symmetries for Riccati equations. We retrieve and generalize results given in [21] to determine their Lie symmetries. Subsequently, we propose their generalization to the realm of split-complex and Study numbers [32,36]: the *Cayley–Klein Riccati equation*. We prove that the Cayley–Klein Riccati equation with real t-dependent coefficients is a Lie system admitting a Vessiot–Guldberg Lie algebra isomorphic to $\mathfrak{sl}(2, \mathbb{R})$, namely an $\mathfrak{sl}(2, \mathbb{R})$-*Lie system* [30], and analyze its Lie symmetries with our methods. Next, we study quaternionic Riccati equations with real t-dependent coefficients. We prove that they are $\mathfrak{sl}(2, \mathbb{R})$-Lie systems and give a Lie system describing some of its Lie symmetries. Since all of the above mentioned Lie systems are $\mathfrak{sl}(2, \mathbb{R})$-Lie systems, our techniques enable us to obtain a common equation describing Lie symmetries for all of them.

As a particular example, we generalize results concerning Lie symmetries for generalized and classical Darboux–Brioschi–Halphen systems [28]. Apart from the interest in these systems due to their applications [14], our example illustrates how Lie systems can be employed to investigate autonomous systems for the first time. Autonomous systems of first-order ordinary differential equations are Lie systems associated with a one-dimensional Vessiot–Guldberg Lie algebra. This implies that most of the methods to analyze Lie systems cannot be applied, as they only work properly for Vessiot–Guldberg Lie algebras of dimension two or higher (see [12]). We show that this problem can be avoided by studying autonomous systems as Lie systems related to Vessiot–Guldberg Lie algebras of dimension greater than one. Although finding such Lie algebras can be difficult, we succeeded in determining some of them for certain relevant Lie systems.

Subsequently, we illustrate how our results can be used to study HODE Lie systems [8]. Such systems are systems of higher-order differential equations that can be written as a Lie system by considering them as a first-order system in the usual way, namely by adding extra variables $v \equiv dx/dt$, $a \equiv dv/dt$, etc. Using this fact and the techniques of this work, we obtain *non-local Lie symmetries* for the initial HODE Lie system under study, where by non-local we mean Lie symmetries depending on the successive derivatives of the dependent variables. We also study Lie algebras of Lie point symmetries for HODE Lie systems, i.e., Lie symmetries that only depend on their independent and dependent variables.

Finally, we extend our results to PDE Lie systems [9]. Similarly to Lie systems, PDE Lie systems can be associated with a Vessiot–Guldberg Lie algebra. Also, they can be related to a Lie algebra of Lie symmetries of a certain type which is again determined by the Vessiot–Guldberg Lie algebra of the initial system. As before, this permits us to calculate simultaneously Lie symmetries for all PDE Lie systems possessing isomorphic Vessiot–Guldberg Lie algebras by solving another PDE Lie system. To illustrate our results, we analyze PDE Lie systems of relevance in physics and mathematics, e.g., multidimensional Riccati equations and PDEs describing certain flat connection forms [20]. It is remarkable that the literature on PDE Lie systems is scarce (specially concerning their uses) and this paper contributes to enlarge their applications and to understand their properties [12].

The structure of the paper goes as follows. In Section 2, we briefly review the concept of Lie systems and PDE Lie systems. Section 3 is devoted to the analysis of Lie symmetries for Lie systems. In Section 4 we provide methods to build different types of Lie algebras of Lie symmetries for Lie systems. Subsequently, we apply our results to Lie systems and HODE Lie systems of relevance in Section 5. We extend our methods to PDE Lie systems in Sections 6 and 7. In particular, we calculate Lie symmetries for PDE Lie systems with the same symmetry system and physical and mathematical interest. In Section 8, we summarize our results and we describe our plans for future research.

2. Fundamentals on Lie systems

For simplicity, we hereafter assume all geometric structures to be real, smooth and globally defined. In this way, we highlight the key points of our presentation by omitting the analysis of minor technical problems (we refer to [9,12] for additional details). Subsequently, N is assumed to be a real n-dimensional manifold with a local coordinate system $\{x_1, \ldots, x_n\}$.

Let $\tau\colon TN \to N$ be the natural projection of the tangent bundle TN onto N and let $\pi_2 : (t, x) \in \mathbb{R} \times N \mapsto x \in N$ be a projection. A *t-dependent vector field* X on N is a map $X : (t, x) \in \mathbb{R} \times N \mapsto X(t, x) \in TN$ such that $\tau \circ X = \pi_2$. This condition implies that every t-dependent vector field X on N can be considered as a family $\{X_t\}_{t\in\mathbb{R}}$ of vector fields $X_t\colon x \in N \mapsto X_t(x) \equiv X(t, x) \in T_xN \subset TN$ on N and vice versa. We call $\Gamma(TN)$ the $C^\infty(N)$-module of sections of the tangent bundle (TN, N, τ).

We call *integral curve* of X an integral curve $\gamma : s \in \mathbb{R} \mapsto (t(s), x(s)) \in \mathbb{R} \times N$ of the *suspension* of X, i.e., the vector field

$$\begin{aligned} \bar{X} : \mathbb{R} \times N &\longrightarrow T(\mathbb{R} \times N) \simeq T\mathbb{R} \oplus TN \\ (t, x) &\mapsto \frac{\partial}{\partial t} + X(t, x). \end{aligned} \tag{2.1}$$