Applied Mathematics and Computation 273 (2016) 453–464

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Hyperbolic Pascal triangles

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ARTICLE INFO

MSC:
11B99
05A10

Keywords:
Pascal triangle
Regular mosaics on hyperbolic plane
Hyperbolic planar tessellations

ABSTRACT

In this paper, we introduce a new generalization of Pascal's triangle. The new object is called the hyperbolic Pascal triangle since the mathematical background goes back to regular mosaics on the hyperbolic plane. We precisely describe the procedure of how to obtain a given type of hyperbolic Pascal triangle from a mosaic. Then we study certain quantitative properties such as the number, the sum, and the alternating sum of the elements of a row. Moreover, the pattern of the rows, and the appearance of some binary recurrences in a fixed hyperbolic triangle are investigated.

1. Introduction

There exist several variations of Pascal's arithmetic triangle (see, for instance, [2] or [3]). This study provides a new generalization. The innovation is to extend the link between the infinite graph corresponding to the regular square Euclidean mosaic and classical Pascal's triangle to the hyperbolic plane, which contains infinitely many types of regular mosaics. The purpose of this paper is to describe this generalization and examine certain properties of the hyperbolic Pascal triangles. From one point we will focus on only one regular square mosaic given by the pair $\{4, q\}$, $q \geq 5$. Later we will specialize in the case $q = 5$. We remark, that in this paper the terminology Pascal's triangle is used only for Blaise Pascal's original triangle, while the expression Pascal triangle (without apostrophe) means a variation or generalization of Pascal's triangle.

Let p and q denote two positive integers satisfying $p \geq 3$, $q \geq 3$, and consider the regular mosaic characterized by the Schläfli's symbol $\{p, q\}$. The parameters indicate that for regular p-gons of cardinality exactly q meet at each vertex. If $(p-2)(q-2) = 4$, the mosaic belongs to the Euclidean plane, if $(p-2)(q-2) > 4$ or $(p-2)(q-2) < 4$, the mosaic is located on the hyperbolic plane or on the sphere, respectively (see, for instance [4]). From now on we exclude the case $(p-2)(q-2) < 4$, since we intend to deal only with infinite mosaics.

The vertices and the edges of a given mosaic determine an infinite graph $\mathcal{G}$. Fix an arbitrary vertex, say V_0. Obviously, the q-fold rotation symmetry around V_0 and q different mirror symmetries of the mosaic induce a q-fold rotation symmetry and mirror symmetries on the graph. Consequently, in accordance with the object of examination, it is sufficient to consider only a certain subgraph $\mathcal{G}'$ of $\mathcal{G}$. In order to introduce hyperbolic Pascal triangles, now we define our own $\mathcal{G}'$ precisely.

Restrict the mosaic to an unbounded part $\mathcal{P}$ of itself by defining first the border of $\mathcal{P}$, which contains regular p-gons as follows. The border splits the mosaic into two more parts, and the part which contains just one symmetric line among the q different symmetric lines through the vertex V_0, together with the border gives $\mathcal{P}$.

Taking a regular p-gon which fits to V_0, we consider it as the $0th$ (base) cell of the border. Beginning from V_0, figure the edges of the base p-gon by $1, 2, \ldots, p$ anti-clockwise (see Fig. 1). For the case of even and odd p, let first $p = 2k$ $(k \geq 2)$. Mirror the base

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http://dx.doi.org/10.1016/j.amc.2015.10.001
0096-3003/

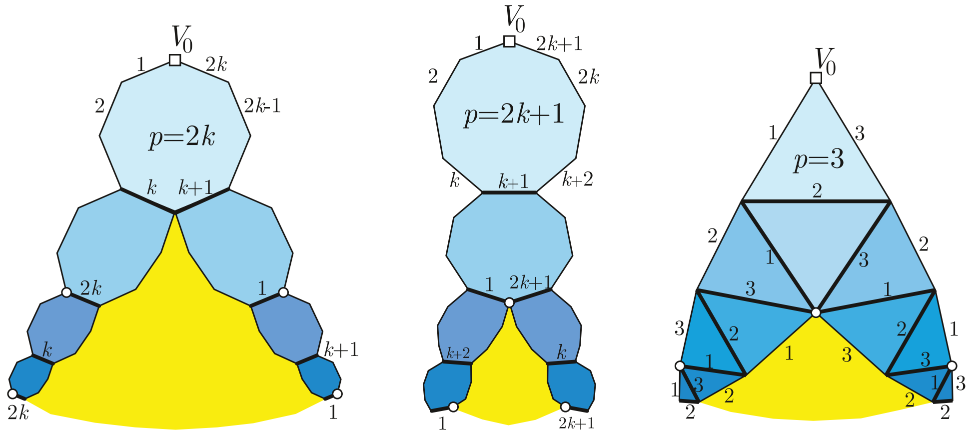

Fig. 1. Construction of the border of $\mathcal{P}$. (For interpretation of the references to colour in the text, the reader is referred to the web version of this article.)

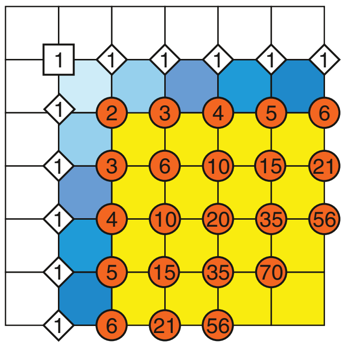

Fig. 2. Pascal's triangle on the Euclidean mosaic $\{4, 4\}$.

cell across the edge k to obtain the first cell of the left hand side part of the border. We assume that the reflection transmits the figuring of the edges as well, for instance the edge k is common of the 0th and 1st cells. Then to gain the second cell we reflect the first p-gon through the edge $2k$ of the first cell. And so on, we always mirror the ℓth cell through the edge k or $2k$ to obtain the $(\ell + 1)$th element of the left hand side of the border. Getting back to V_0 and cell 0, one can get the right hand side part of the border in a similar way by reflecting the appropriate elements across the edges $k + 1$ and 1, alternately. If $p = 2k + 1 \geq 5$, first we reflect the base cell across the edge $k + 1$, and after then we can get the border in an analogous way to the case of even p if we apply reflections through the edges 1, $k + 2$ (LHS) and $2k + 1$, k (RHS), alternately in both cases. If $p = 3$, then after reflecting the base cell of the mosaic across the edge 2, we construct the LHS and RHS borders by consecutive reflections across edges $1, 3, 2, 1, 3, 2, \ldots,$ and edges $3, 1, 2, 3, 1, 2, \ldots,$ respectively. Fig. 1 shows the construction of the border of $\mathcal{P}$ (and $\mathcal{P}$ itself when one joins the blue and yellow parts-dark and light in black and white version), first if p is even, second if $p \geq 5$ is odd, and then when $p = 3$. Moreover, Fig. 4 illustrates the case $\{4, 5\}$.

Since $\mathcal{P}$ is a definite part of the mosaic, it eliminates a subgraph $\mathcal{G}_{\mathcal{P}}$ in the graph $\mathcal{G}$ of the whole mosaic. In the sequel, we always consider only this graph $\mathcal{G}_{\mathcal{P}}$. Assume that all edges have unit length. The distance of an arbitrary vertex V and V_0 is the length of the shortest path between them. It is clear, that the shortest path from V_0 to any vertex of the outer boundary of $\mathcal{P}$ is unique, and passing through on the outer boundary itself. Inside of $\mathcal{P}$ there are always at least two shortest path from V_0 to any V. Let us label all vertices V of $\mathcal{G}_{\mathcal{P}}$ by the number of distinct shortest paths from V_0 to V. As the first illustration take the Euclidean squared mosaic (i.e. $\{p, q\} = \{4, 4\}$). Then the subgraph $\mathcal{G}_{\mathcal{P}}$ with its labelling returns with Pascal's original triangle (see Fig. 2).

The other two Euclidean regular mosaics have no great interest since they are also associated with Pascal's triangle (see Fig. 3). In case of $\{3, 6\}$ the appearance is direct, while $\{6, 3\}$ displays all rows twice. But any hyperbolic mosaic $\{p, q\}$, in the manner we have just described, leads to a so-called hyperbolic Pascal triangle (see the mosaic $\{4, 5\}$, Fig. 4). Thus the Euclidean mosaics and the hyperbolic mosaics together, based on the idea above, provide a new generalization of Pascal's triangle. In the next section, we narrow the spectrum of the investigations by concentrating only on the mosaics having Schläfli's symbol $\{4, q\}$. We do this to demonstrate the most spectacular class of hyperbolic Pascal triangles, when the connection to Pascal's triangle is the most obvious.